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ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1934 DECEMBER 14

Prof. F. J. M. Stratton, President, in the Chair.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

- Arthur Beer, Ph.D., Solar Physics Observatory, Cambridge (proposed by F. J. M. Stratton) ;
William Gee Bickley, D.Sc., 141 Byrne Road, London, S.W. 12 (proposed by L. J. Comrie) ;
William Fitkin Muckle, A.M.Inst.C.E., Asst. Civil Engineer, Port of London Authority, Haddon, Shepherds Lane, Dartford Heath, Kent (proposed by A. Fowler) ;
Mrs. Vera Reade, Khandala, 12 Gillian Park Road, Sutton, Surrey (proposed by F. Robbins) ;
Sadanand Waman Shiveshwarkar, B.A., Sidney Sussex College, Cambridge (proposed by W. M. Smart) ; and
John Davis Williams, Steward Observatory, Tucson, Arizona, U.S.A. (proposed by E. F. Carpenter).

Seventy-five presents were announced as having been received since the last meeting, including :—

British Association Mathematical Tables, vol. 4 (presented by the British Association Tables Committee).

Mr. D. H. Sadler and Dr. J. C. P. Miller were appointed auditors of the Treasurer's accounts for 1934.

A POSSIBLE IDENTIFICATION OF NOVA VULPECULÆ (1670).

W. H. Steavenson.

The nova discovered in Vulpecula by Anthelm in 1670 has never been certainly identified. The late J. R. Hind, working up some rough observations given in Lemonnier's *Histoire Céleste*, deduced for the nova the position $\alpha = 19^{\text{h}} 41^{\text{m}} 26^{\text{s}}.8$, $\delta = +26^{\circ} 56' 36''$ (Equinox of 1850). The same observer found, in 1861, near this place a star of about the eleventh magnitude, which he considered to be variable in light and nebulous in appearance (*M.N.*, 21, 231, 1861).

The late E. E. Barnard, observing with the 40-inch Yerkes refractor in 1913, examined Hind's star on several nights without finding anything

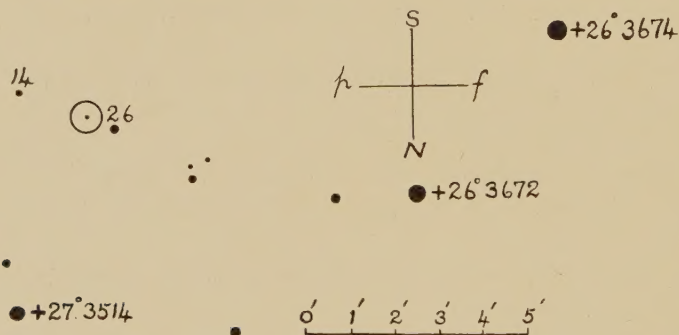


FIG. 1.

unusual in its light or appearance ; nor am I aware of any observations by other observers which would tend to confirm the peculiarities noted by Hind.

Details of Barnard's observations were published by him in *A.N.*, 4753, 1, where he gives a chart of the faint stars visible to him in the immediate neighbourhood of the nova's reputed position. This chart, which represents a circular field of 5.5 diameter, probably shows everything seen by Barnard, since it includes several stars of magnitude 16.5 or fainter, even towards the boundaries of the field ; and it seems unlikely that he would have missed any object substantially brighter than this, especially nearer to the probable place of the nova. I was therefore surprised, when examining the field with the $20\frac{1}{2}$ -inch Hindle reflector on 1934 July 5, to see a star of approximately $15^{\text{m}}.4$ within about half a minute of arc of Hind's estimated position, and in a place where nothing is shown on Barnard's chart.

At my request Dr. Knox-Shaw very kindly undertook to have the field photographed with the 24-inch telescope of the Radcliffe Observatory, and two plates were secured by Mr. H. G. Scott Barrett, the first on 1934 July 10 and the second on 1934 October 29. Both plates clearly confirm the existence of the small star observed by me, and further show that its image is unlike

those of other stars of about the same magnitude, being far more sharply defined. This would seem to point to some spectral peculiarity, brought out by the particular colour-correction of the 24-inch lens. In addition, the photographs indicate a definite variability in the star's light. It is difficult to make good comparisons with other stars, owing to the peculiar character of the image, but the plate of October 29 would seem to indicate a rise of about $0^m.4$ as compared with that of July 10.

Hind's computed place for the nova, brought up to the epoch of Barnard's observations (1913.0), is $19^h 44^m 1^s.6$, $+27^\circ 5' 43''$. The place of the star actually observed by Hind was measured by Barnard and found by him to be $19^h 44^m 6^s.3$, $+27^\circ 6' 13''.1$ (1913.0). It is numbered 26 on the chart of the latter observer, and is identical with No. 816 of the Greenwich Catalogue of 1872.

It will be noted that this star is distant $+4^s.7$ in R.A. and $+30''.1$ in Dec. from the position computed for the nova by Hind. The star observed

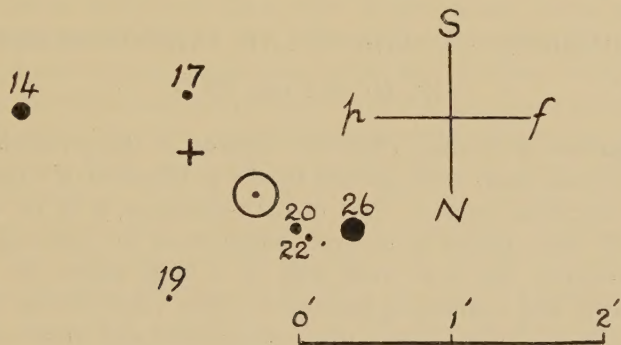


FIG. 2.

by me in 1934 July precedes Hind's star by $2^s.9$, and is $12''.6$ south of it, so that its position for 1913.0 is $19^h 44^m 3^s.4$, $+27^\circ 6' 0''.5$. It is thus much nearer to the computed place, from which it is distant only $+1^s.8$ in R.A. and $+17''.5$ in Dec., or, in arc of a great circle, about $30''$ only.

If we bear in mind the probable error of the original rough observations of 1670, the existence of a variable star with a peculiar spectrum so close to the supposed position of the nova would seem to suggest a strong probability of its identity with this long lost body.

In view of its presumed invisibility in the 40-inch refractor in 1913 it is reasonable to expect that the small star will not remain indefinitely at its present brightness, and it therefore seems desirable that it should be observed and photographed as soon as possible with large instruments before it becomes once more a faint object.

In figs. 1 and 2 a small circle surrounds the star observed in 1934. The first figure shows the approximate position of this star in relation to the three nearest B.D. stars. The second figure, which is on a larger scale, shows a few of the faint stars in the immediate vicinity of the place computed by Hind, which is indicated by a cross. The numbers against the stars

are those of Barnard (*loc. cit.*). It is to be noted that his 17 is rather nearer to Hind's position than is the star observed by me. But, apart from this, it seems to exhibit no peculiarities such as would point to the probability of its being the nova. My rough estimates of the magnitudes of the four faintest stars are as follows: $20 = 13.7$, $17 = 14.2$, $22 = 15.0$, $19 = 15.2$. I was unable to see the faint companion to 17, shown on Barnard's chart. His second companion to 20, north following 22, was occasionally glimpsed, and estimated to be of magnitude $15.5 \pm$. It must be a reddish star, as it is not shown on the Radcliffe plates, which otherwise exhibit all that Barnard saw. The star within the circle was observed on 1934 July 5, 6, 9, 10, 14 and on August 3. On all these nights it was estimated to be about $0^m.2$ fainter than No. 19.

THEORIES OF THE SOLAR CHROMOSPHERE.

W. H. McCrea, *Ph.D.*

1. *Comparison of Existing Theories.*—Interest in the problem of the solar chromosphere has lately been revived by the publication of Chandrasekhar's important theoretical work.* Part of its importance is to be found in the circumstance that, instead of entirely supplanting all previous theories of the chromosphere, the new work may to a large extent be regarded as extending them and combining features in them which before had seemed, if not mutually exclusive, at any rate unconnected and unnecessary to each other. The present remarks are therefore an attempt to examine the relation between existing investigations and to state some of the problems still requiring solution.

The first of the theories of the chromosphere was Milne's theory of selective radiative support, and Chandrasekhar's theory is properly regarded as an extension of this. Milne's theory is still the only one which is complete in itself, containing no extraneous hypotheses, and the only one in which the mathematical treatment is also effectively complete. It met with difficulties, however, in explaining the presence in the chromosphere of elements whose principal lines do not lie in the intense part of the solar spectrum, in accounting for the existence of high velocities of chromospheric atoms, and in explaining why the chromospheric density gradient is apparently exponential.

In order to meet some of these difficulties the "turbulence theory" was put forward. It accepted the observed existence of large velocities in the chromosphere, and by seeking roughly to describe them by use of an "apparent temperature" tried to infer their effect upon the density gradient. In the light of Chandrasekhar's work it now appears that this "theory" may have been correct in recognizing that the problem of the high velocities

* S. Chandrasekhar, *M.N.*, 94, 14-35, 726-737, 1934.

and the problem of the density gradient are fundamentally the same. Beyond this it was in fact scarcely a theory at all, but rather, as Chandrasekhar would urge, a re-description of the observed facts.

Chandrasekhar's theory is an attempt to go further into the underlying causes of the phenomena, and may be summarized in a few words. He postulates that in a principal absorption line of a chromospheric atomic species the intensity of radiation at the base of the chromosphere varies periodically over the Sun's surface, with a mean value equal to that required just to support the atoms by selective radiation pressure against gravity. He then works out the trajectories of individual atoms under these conditions. In particular he shows the existence of *periodic* orbits. This is easily understood physically; for if from a point in the atmosphere we project an atom obliquely downwards towards a region of maximum intensity, the excess upward force due to radiation pressure tends to deflect its trajectory upwards, and then if it arrives above a region of minimum intensity the excess downward force due to gravitation tends to deflect it downwards once again, and so on. As we should expect, it is possible to choose the circumstances of projection so that the trajectory becomes periodic with the same period in horizontal distance as that possessed by the radiation intensity. If we neglect the perturbations of such trajectories by collisions or other agencies, then any arbitrary assemblage of them will constitute a possible permanent "chromosphere." We can in fact imagine quite fantastic cases where this "atmosphere" is altogether detached from the Sun itself. But Chandrasekhar shows that if for definiteness we select such an assemblage of orbits that the velocity is a one-valued function of position, with suitable boundary conditions, then the density gradient will closely mimic an exponential law of the observed character, and, moreover, the mean velocity will be of the observed order of magnitude. This depends only on choosing the period and amplitude of the intensity fluctuation to be of the order indicated by examination of spectroheliograms.

2. *Necessary Extensions of the Theory.*—It is clear that there are many points at which the theory requires amplification before it can be regarded as fully tested. The most obvious is that all the other possible trajectories, besides the periodic ones, should be taken into account. If atoms are projected from the top of the reversing layer then, under the forces contemplated by Chandrasekhar, they will describe a variety of orbits, some going off to infinity and others turning back and re-entering the reversing layer. In fact, if there is no further action, the aggregate of all these orbits, and not the periodic ones, will constitute the chromosphere. For an atom leaving the reversing layer cannot have the right circumstances of projection to describe a periodic orbit which does not again strike that layer. Therefore Chandrasekhar's theory implicitly assumes some further mechanism by which atoms, as he says, "feel their way" towards the periodic orbits, with presumably a reverse process to maintain a steady state. However this is achieved, the steady state of the chromosphere in any neighbourhood must be maintained on this view by trajectories of all classes, periodic and non-periodic, traversing it.

Such a picture would be nearer to that imagined in the turbulence "theory" than is Chandrasekhar's first discussion in which he takes the velocities to be unique functions of position. It is of course different from the ordinary turbulence of hydromechanics since the velocities belong to individual atoms rather than to groups of atoms. A significant approximation might still be got, however, by describing the aggregate of velocities by the "apparent temperature" which would give the same mean velocity, even though there is no *a priori* reason to suppose they will be distributed precisely according to a Maxwell law. The density distribution could not, however, be deduced by treating the atmosphere as a gas of this temperature in *hydrostatic* equilibrium. For it contains always particles moving out towards infinity, but not the complementary ones moving in from infinity to preserve (macroscopically) the hydrostatic state. Even if any modified calculation on these lines were possible, it would still have to be remembered also that the gas is everywhere subject to radiation pressure as well as to gravitation, which alone was taken into account in the turbulence theory, so that by whatever mathematical means it is handled the theory is always fundamentally one of radiative support.

3. *Use of "Apparent Temperature."*—Nevertheless, in spite of the difficulties just mentioned, it is instructive to recall that the method of trajectories and the use of a temperature can in suitable cases lead to precisely the same result. For consider the case of a gas in hydrostatic equilibrium at temperature T in a field of uniform force mg per atom of mass m . Then the simple hydrostatic calculation for an isothermal atmosphere gives a density law $\rho \propto e^{-mgx/kT}$, where x is measured antiparallel to the field from some fixed plane.

To treat the problem by trajectories we suppose the atoms at this fixed plane to have a Maxwell velocity distribution, and neglect collisions in $x > 0$. Then the number of atoms leaving unit area of the plane per unit time with x -velocity-component in $(u, u + du)$ is $au e^{-mu^2/2kT} du$, where a is constant. These atoms describe parabolic orbits under the given field of force, and their velocities at distance x are in $(v, v + dv)$, where $v^2 = u^2 - 2gx$, $v dv = u du$. Therefore their density is $au e^{-mu^2/2kT} du / v = a(e^{-mv^2/2kT} dv) e^{-mgx/kT}$. Integrating over v in $0 < v < \infty$, which includes all atoms which reach distance x , we again get the law $\rho \propto e^{-mgx/kT}$. (The outward and inward moving particles make equal contributions to ρ .) Thus *the aggregate of all the parabolic trajectories gives just the same density law as that given by hydrostatics*. This is probably a familiar example, and of course one expects the result to be a property of the assumed Maxwell distribution of initial velocities, since otherwise the density law would be affected by the occurrence of collisions in the gas, and so would depend upon the absolute value of the density, which would be contrary to experience. Actually it was this example which originally suggested the use of an "apparent temperature" in the case of turbulence.

4. *Present State of the Theory.*—Chandrasekhar's theory is a great advance upon anything like the turbulence hypothesis, in that it gives a reason for the existence of large velocities of just the observed order of

magnitude. This depends upon the assumption of the intensity fluctuations, which is justified by observation, but which it is outside the scope of the theory to explain. As a matter of fact Rosseland * has suggested that they in turn may be due to turbulence in the solar atmosphere (photosphere). The particular feature which does, however, invite further explanation is why the residual intensity in the principal lines of a chromospheric atom should possess a mean value (I_0) equal to, or nearly equal to, that required for support of the atom by radiation pressure against gravity. Milne showed, as Chandrasekhar points out, that the residual flux on his theory should correspond just to I_0 , but if this result can be applied here it would show that the residual flux *everywhere*, and not merely its average value, should correspond to I_0 .

Finally, there is the still vexed problem of the hydrogen chromosphere. The well-known difficulty in this case of any theory depending on radiation pressure is the absence of any principal line in the intense part of the spectrum. Nevertheless Chandrasekhar considers that his theory "gives a clue" as to how a high-level hydrogen chromosphere could be set up. For hydrogen atoms excited into the second quantized state, and so capable of absorbing the Balmer lines, will *tend* to form an atmosphere like Ca^+ , but as soon as one of these atoms returns to the normal state, or becomes ionized, it is on this view lost to the chromosphere. Chandrasekhar describes the state of affairs by saying that "such chromospheres should perpetually be in a state of being formed." It is a matter for detailed calculation to see how far such formation can proceed. The difficulties previously pointed out by me † have still to be met. In particular I found for the base of the chromosphere a number of free electrons present of the order of 10^{10} per c.c. as compared with something like 10^6 Ca^+ ions per c.c. Rosseland ‡ has independently found a number of free electrons of the order 10^8 per c.c. at the top of the chromosphere. These must be accompanied by a similar number of positive ions, and, since this number is large compared with the number of Ca^+ ions which furnish the most favourable case for support by radiation pressure, one is compelled to look for a further mechanism of support for such particles. If these ions, which I suggested would probably be hydrogen ions, could by any means be given the requisite velocities, of the order of those found by Chandrasekhar, they could set up a density gradient of the sort studied in the turbulence theory. This might be accomplished by collisions with fast-moving Ca^+ ions, or others that are subject to radiation pressure. On the other hand, some entirely different mechanism such as that suggested by Rosseland ‡ may be at work.

This latter theory, it should be pointed out, opens up quite new fields of possibility. For, as we have seen, if Chandrasekhar's views are correct, they can be regarded as incorporating various essential features of previous work. On the other hand, if Rosseland's views prove correct in the case

* S. Rosseland, *M.N.*, 89, 49-53, 1929.

† W. H. McCrea, *M.N.*, 89, 483-497, 1929.

‡ S. Rosseland, *University Observatory, Oslo, Pub. No. 5*, pp. 1-37 (see p. 24),

of the chromosphere, then they will necessarily supersede, rather than incorporate, the earlier ones, but in doing so will offer solutions of a wider range of problems, including some of those connected with the corona. The present state of observation and theory seems to offer no short cut to a decision between the different sets of ideas, and the only course open to investigators seems to be a full exploration of the theoretical consequences of each set, and a full comparison of the results with all available observational data on all relevant phenomena.

As far as one can sum up the present situation it seems to be that Milne's theory of selective radiative support in Chandrasekhar's modified form gives an adequate account of the Ca^+ chromosphere. But we have to suspend judgment for gases like hydrogen and helium until their cases have been worked out in detail. Until that is done, or another theory is established, we cannot say even whether radiation pressure provides the normal means of chromospheric support, with hydrogen and helium occupying somewhat exceptional positions, or whether, on the other hand, normal chromospheric support comes from quite a different agency with a few gases like Ca^+ and Sr^+ showing additional peculiarities owing to their happening to be subject in addition to radiation pressure.*

Finally, I should like to say that, though I do not venture to suppose that Dr. Chandrasekhar will share all my views, yet I am greatly indebted to him for discussions which have helped me to an understanding of his theory.

Imperial College of Science :
1934 December.

RADIATION-PRESSURE IN AN EXPANDING NEBULA.

H. Zanstra, Ph.D.

1. *Introduction.*—The present paper deals with radiative equilibrium, and more in particular with radiation-pressure, in an expanding nebula. For a stationary nebula this problem was treated by Ambarzumian.† He also treated the case of an expanding nebula consisting of a spherical shell, all parts of which have the same outward radial velocity with respect to the central star.‡ One of the main outcomes of Ambarzumian's work is the predominant rôle which radiation-pressure due to the first line of the Lyman series L_α assumes in these cases, this force being very much larger than the force of gravity of the central star in the case of a planetary nebula.

* I pointed out (*loc. cit.*) that the presumably small degree of ionisation of Ca^+ to Ca^{++} could better be understood if an excess of free electrons from some other source were also present.

† V. A. Ambarzumian, *M.N.*, 93, 50, 1932.

‡ V. A. Ambarzumian, *Bull. de l'Obs. centr. à Poulkovo*, 13, 3, No. 114, 1933.

However, there are grounds for supposing the actual expansion to be such that the radial velocity of the different portions of the envelope increases with the distance from the central star. This is the type of expansion we are dealing with in the present paper. Our main aim is to investigate how far the radiation-pressure is cut down by an expansion of this type. The result will be that the radiation-pressure is very much lessened, provided that the "thermal" velocity w is small compared with the range of velocity v_L over the luminous portion of the nebula; in fact, it is of the order $(w/v_L)^2$ as compared with the radiation-pressure in the stationary case.

If a planetary nebula originates by a catastrophe near the surface of the star, an original range of velocity is very likely to occur. Subsequent action of gravity results in a much larger spreading of velocities than originally present,* whereas radiation-pressure † tends to decelerate the inner portions and accelerate the outer portions, thus making the effect more pronounced.

2. *Radiation-pressure Due to the Continuous Spectrum. Line Absorption Coefficient.*—The formulæ given in this section are applicable to expanding as well as non-expanding nebulae.

We make the usual assumptions and simplifications of the theory of nebular luminosity. ‡ The nebula is considered to consist mainly of hydrogen, with an admixture of ions like O^+ and O^{++} , and is situated at a distance r from the exciting star. The latter is represented by a black body of temperature T and radius R . The hydrogen is then able to absorb the quanta L_c of the ultra-violet stellar radiation beyond the head of the Lyman series, frequency ν_0 , which we consider to be incident perpendicular to the nebula. The number πS_c of such quanta incident per cm^2 per sec. is then given by

$$S_c = \frac{2R^2}{r^2} \cdot \frac{k^3 T^3}{c^2 h^3} \int_{x_0}^{\infty} \frac{x^2}{e^x - 1} dx, \quad x_0 = \frac{h\nu_0}{kT}. \quad (1)$$

The nebula is considered to be a parallel slab of thickness small compared with r , which means either that we consider a small portion of a rather thin spherical shell of nebulosity surrounding the star, or an isolated patch of nebulosity at a distance r § (see fig. 1).

For the nebular H -atoms, the atomic absorption coefficient a_c of the continuous spectrum beyond the head of the Lyman series depends on the frequency, but with Ambarzumian we shall consider it constant and equal to its value at the head, which, according to Sugiura,|| is

$$a_c = 0.54 \times 10^{-17}. \quad (2)$$

If C_a be the number of neutral H -atoms per c.c. in the nebula at a distance

* H. Zanstra, *M.N.*, **93**, 131, 1932.

† V. A. Ambarzumian, *Bull. de l'Obs. centr. à Poulkovo*, **13**, 3, No. 114, pp. 17, 18.

‡ Cf. H. Zanstra, *Publ. D.A.O. (Victoria)*, **4**, 209, 1931; *Zeitschr. f. Astrophys.* **2**, 1, 1931.

§ If, in the latter case, the incident radiation makes an angle ϕ with the normal, one may use all of the following results with $\tau \sec \phi$ for τ and $S_c \cos \phi$ for S_c .

|| Cf. H. Zanstra, *Publ. D.A.O. (Victoria)*, **4**, 255, 256, 1931.

ξ measured from the inner boundary outwards, that is away from the star, we have for the optical depth τ for the continuous spectrum,

$$\tau = \int_0^{\xi} \alpha_c C_a d\xi. \quad (3)$$

Ambarzumian, making use of mathematical methods developed by Milne,* has considered the re-emission of L_c -quanta. For the sake of simplicity we shall, however, neglect this re-emission; if recombination

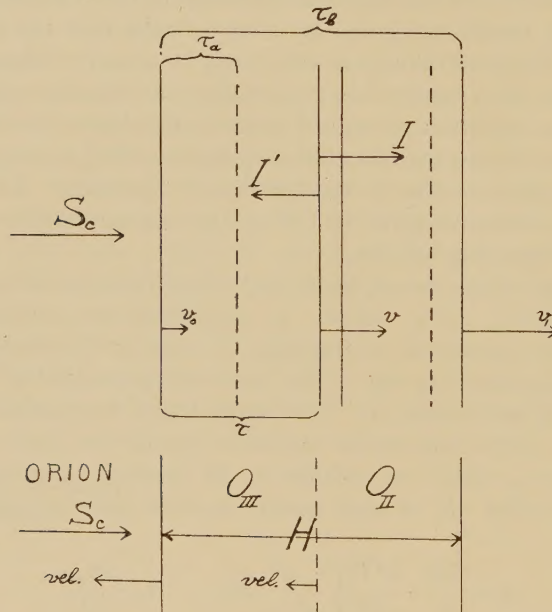


FIG. 1.—Above: expanding nebula. Below: Orion nebula.

took place on the higher levels of the H -atom, this would be an exact treatment. The outward flux πF_c of quanta L_c is then given by the following equation (4a). The force per cm^2 on $d\tau$ due to radiation-pressure of L_c is obtained by multiplying the energy flux by $\frac{d\tau}{c}$,† hence

$$dP_{cr} = \frac{h\nu_0}{c} d\tau \cdot \pi F_c, \quad (4)$$

$$F_c = S_c e^{-\tau} \sim S_c, \quad (4a)$$

where the sign $\sim$ means order of magnitude.

We have put the frequency of the continuous spectrum equal to its value at the head ν_0 , which, like (2), only constitutes a very rough approximation; in fact, equation (4) is hardly better than an order of magnitude.

We now consider the re-emission of the line spectrum by the nebula. For each quantum L_c absorbed a recombination takes place, and, since we

* E. A. Milne, *Zeitschr. f. Astrophys.*, **1**, 98, 1930.

† E. A. Milne, *Handbuch der Astrophysik*, **3**, chap. ii, p. 100.

neglect the L_c -re-emission, each recombination results in the re-emission of one of the quanta of the Lyman series, L_α , L_β , L_γ , etc. By processes of the type $L_\beta \rightarrow L_\alpha + H_\alpha$, $L_\gamma \rightarrow L_\alpha + H_\beta$, etc., these quanta, in so far as they are not L_α , will be transformed into this line after very few re-emissions.* In calculating the approximate radiation-pressure due to the Lyman series, we are entitled to treat all quanta as L_α , a line which predominates even at the first re-emission. In any case, this treatment gives an upper limit to the force of radiation-pressure, since the absorption coefficient of the higher lines is smaller than for L_α . The result is therefore that *the number of L_α -quanta produced per unit volume may be considered to be approximately equal to the number of L_c -quanta absorbed.*

In each volume element, taken by itself, the H -atoms have a certain distribution of velocity, which is presumably of the Maxwellian type. We shall denote this velocity, or rather one of its components, *e.g.* that perpendicular to the nebular slab, by W , and call it the "thermal velocity," which is merely a convenient term suggested by the analogy to the motion in actual thermal equilibrium. We further introduce the simplification that W is evenly distributed between $-w$ and $+w$, where $2w$ is the thermal width in km./sec., which we consider constant throughout the nebula, though it is realised that this may not be the actual situation. Then

$$\frac{W}{c} = \frac{\nu - \nu_\alpha}{\nu_\alpha}, \quad \frac{2w}{c} = \frac{\Delta}{\nu_\alpha}, \quad (5)$$

where ν_α is the undisplaced frequency of L_α , ν the frequency considered and Δ the thermal width in frequency units.

The line absorption coefficient is thus constant within the frequency interval Δ or velocity interval $2w$, and given by †

$$\alpha_l = f \frac{\pi e^2}{mc} \frac{1}{\Delta} = \frac{0.668 \times 10^{-12}}{w}, \quad (6)$$

for

$$\nu_\alpha + \frac{1}{2}\Delta > \nu > \nu_\alpha - \frac{1}{2}\Delta \quad \text{or} \quad w > W > -w, \quad (6a)$$

and zero outside this interval, it being understood that w should be large compared with the natural width of radiation damping. The numerical value in (6) is that corresponding to $f = 0.416$.‡

We further introduce the ratio

$$u = \frac{\tau}{\sigma} = \frac{\alpha_c}{\alpha_l} = 0.81 \times 10^{-5} w, \quad (7)$$

as follows from (2) and (6), the quantity σ being defined by

$$\sigma = \int_0^\xi \alpha_l C_a d\xi. \quad (8)$$

* *Astrophys. J.*, 45, 56, 1927.

† Cf. *Publ. D.A.O. (Victoria)*, 4, 255, 1931.

‡ T. Sugiura, *Journal de Physique et Le Radium*, Série IV, 8, 113, 1927.

3. *Radiation-pressure in a Stationary Nebula.*—The formulæ of this section, except (4b), all hold for a stationary nebula only, and, because of this limited applicability, are referred to in Roman figures.

The optical depth for L_α is σ , as defined by (8).

The force of radiation-pressure per cm^2 on $d\sigma$ due to L_α is the product of energy flux and $d\sigma/c$, hence

$$dP_r = \frac{h\nu_\alpha}{c} d\sigma \cdot \pi F, \quad (\text{I})$$

$$F = S_e(1 - e^{-\tau}) \sim S_e, \quad (\text{Ia})$$

where πF represents the flux in quanta L_α per cm^2 per sec. Equation (Ia) has been obtained as follows. We consider the L_c -quanta absorbed to be transformed into L_α on the spot (section 2). If the nebula has radial symmetry and completely surrounds the star, which is the case of internal illumination considered by Milne, these quanta can only escape outwardly, and their flux is given by (Ia).*

Consider, at an optical depth τ , σ , a volume of 1 cm^3 , formed by an area of 1 cm^2 parallel to the nebular slab and a depth of 1 cm . If C_i be the number of H -ions per c.c. and m_H the mass of an H -atom, the mass contained in this volume is $(C_a + C_i)m_H$, the optical depth $\Delta\tau = C_a\alpha_c$, $\Delta\sigma = C_a\alpha_i$. The acceleration a_{cr} due to radiation-pressure by L_c is the force (4) divided by this mass; similarly the acceleration a_r due to radiation-pressure by L_α is the force (I) divided by this mass. Hence

$$a_{cr} = \frac{0.54 \times 10^{-17}}{m_H i} \frac{h\nu_0}{c} \cdot \pi F_c, \quad (4b)$$

$$a_r = \frac{0.668 \times 10^{-12}}{wm_H i} \frac{h\nu_\alpha}{c} \cdot \pi F, \quad (\text{Ib})$$

where $i = 1 + C_i/C_a$, and the values (2) and (6) for α_c and α_i have been substituted.

The central star of a planetary nebula is approximately of absolute bolometric magnitude zero; the radius R , expressed in solar radii $R_\odot$, is then given by $\log(R/R_\odot) = +8.5 - 2 \log T$.† Table I deals with a typical planetary nebula of radius 10,000 astronomical units. In the paper cited above,† the ionisation i , or rather an upper limit to it, was estimated by considering the luminous portion to be a shell of 2000 astronomical units thickness. The concentration of neutral atoms was found to be $C_a = 6.2$ per c.c. The values of the orders of magnitude of a_{cr} and a_r are then obtained from (4b), (Ib) and (1) for $S_c = F_c = F$.

* Ambarzumian has also considered as a second case that of a nebula, all parts of which are moving outwardly with one velocity, which is much larger than w . On account of the Doppler effect the nebula is not illuminated by L_α from the inside, and the L_α -quanta escape outwardly as well as inwardly, their total amount being the same as for a stationary nebula. It is clear that, also in this case, the order of F is S_e (Ia) for the major portion of the nebula.

† *Zeitschr. f. Astrophys.*, 2, 335, 1931.

TABLE I

Planetary of Radius $r_A = 10,000$ Astronomical Units, Central Star of Zero Bolometric Magnitude, $C_a = 6.2$. Accelerations for $F_c = S_c$ and $F = S_c$.

Stellar Temp. $T =$	25,000°	40,000°	100,000°	} in km./sec. per year
Ionisation $i =$	140	260	280	
Order of Accel. due to $L_c : a_{cr} = 2.95 \times 10^{-5}$		4.92×10^{-5}	6.2×10^{-5}	
Order of Accel. due to $L_a : a_r =$	$\frac{2.74}{w}$	$\frac{4.56}{w}$	$\frac{5.72}{w}$	

For other ionisations i and nebular radii r_A the accelerations a_{cr} and a_r are easily obtained by considering the fact that the accelerations are inversely proportional to ir_A^2 , as follows from (4b), (1b) and (1).

Let us pursue the consequences of the assumption that the radiation-pressure in an expanding nebula is of the same order as in a stationary nebula. We then may take for the order of a_r the value for the temperature 40,000° of Table I:

$$a_r = \frac{4.56}{w} \text{ km./sec. in one year,} \quad (\text{II})$$

for $r_A = 10,000$ astronomical units. This is also a lower limit for a_r when r_A is smaller, and holds for the luminous envelope as well as for the outer non-luminous portions within 10,000 astronomical units. Now there is no doubt that a planetary nebula cannot be in equilibrium, and that the material of its envelope originates from the central star.* The velocity gained by matter under radiation-pressure since its expulsion from the star is therefore at least that given by

$$\frac{1}{2}v^2 > a_r r_A = \frac{21.6}{w} r_A (\text{km./sec.})^2, \quad (\text{III})$$

where (II) has been substituted expressed in the proper units.

For planetary nebulae of radii smaller than 10,000 astronomical units, the average line width may reasonably be estimated from the data of Campbell and Moore. The average value of the line width for the 18 nebulae of Table I (*M.N.*, 93, 131, 1932) is $2v_e = 20$ km./sec., assuming for narrow lines, indicated in this table by n , the value of 5 km./sec. This gives an upper limit of 10 km./sec. for the thermal velocity, or $w < 10$ km./sec. Substituting this in (III), we obtain, for a nebula of radius 10,000 astronomical units,

$$v > 208 \text{ km./sec.} \quad (\text{IV})$$

This would lead to lines of a half-width of at least 200 km./sec., and we have just seen that the observed half-width is only 10 km./sec. In the following we hope to show that the force of radiation-pressure in an expanding nebula may be much less than in a stationary nebula with the same w , thus offering a solution of the difficulty.

* Compare the writer's paper in *M.N.*, 93, 132, and Ambarzumian's paper in *Poulkovo Bulletin*, 13, 3, p. 13.

4. *Expanding Nebula*, $v=f(\tau)$.—The whole of section 2, and, in general, all formulæ indicated by arabic figures, hold for the general case, and are therefore also applicable to the case of the expanding nebula we now wish to consider.*

Referring back to equation (8),

$$\sigma = \frac{\tau}{u} = \int_0^{\xi} \alpha_i C_a d\xi; \quad (8)$$

this quantity σ , in an expanding nebula, does not represent a true optical depth, and we may term it the formal optical depth for the line L_a .

Let the velocity of matter in bulk, or mechanical velocity v , referred to the exciting star, be given as a function of τ :

$$v=f(\tau)=f(u\sigma), \quad (9)$$

or its inverse function:

$$\tau=u\sigma=\phi(v), \quad (10)$$

where v is considered positive in the direction away from the exciting star, and is perpendicular to the slab (see fig. 1).

Since the nebula is expanding,

$$\frac{dv}{d\tau} > 0, \quad \frac{dv}{d\sigma} > 0. \quad (10a)$$

We may remark that this condition in any actual case is very likely to hold after a certain length of time, since contraction, unlike expansion, is not a process which can continue indefinitely.

By the term "velocity" we shall always mean the component perpendicular to the slab, unless stated otherwise. The individual velocity of separate atoms, or atomic velocity V , is then the resultant of mechanical velocity v and thermal velocity W , and is given by

$$V=v+W. \quad (11)$$

Since atoms only occur in the interval (6a) for W , the atomic velocities V only occur in the interval:

$$v+w > V > v-w \quad (12)$$

or

$$\tau_a < \tau < \tau_b \quad (13)$$

where

$$\tau_a=u\sigma_a=\phi(V-w), \quad \tau_b=u\sigma_b=\phi(V+w) \quad (13a)$$

because of (10).

The diagram (fig. 2) is a convenient way of representing this, where abscissæ are τ and ordinates V . Of the three curves drawn, the central curve is $V=v=f(\tau)$ (9), and the two extreme curves represent $V=v+w$

* In *B.A.N.*, 7, 7, 1934, August 8, J. Woltjer, jun., has treated the case of scattering through an expanding medium by a method similar in principle to that followed in the present paper.

and $V = v - w$; hence, according to (12), atoms only occur in the region bounded by the two curves. Keeping this in mind, the equations (11), (12), (13) and (13a) may be read off the diagram.

5. *Solution in the V -system for Atoms of Class $V \pm \frac{1}{2}dV$.*—Consider the group of atoms having atomic velocities in the interval $V - \frac{1}{2}dV$ to $V + \frac{1}{2}dV$. These velocities are represented by the narrow horizontal strip parallel to the line marked V in fig. 2, and the corresponding atoms only occur in the

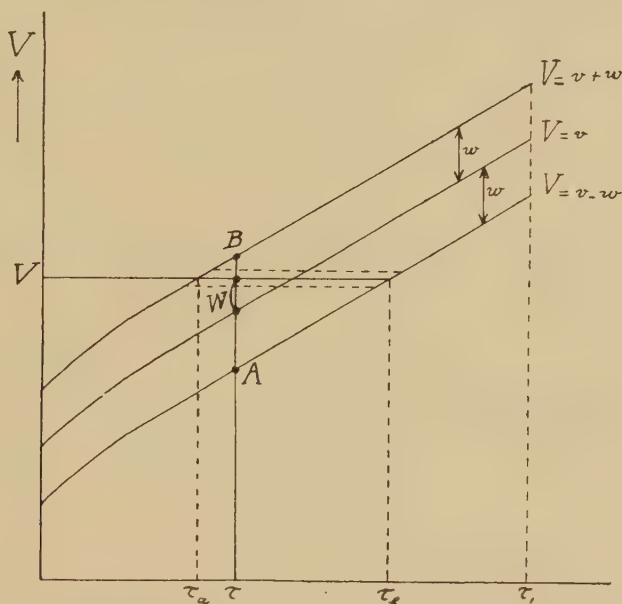


FIG. 2.— V against τ . Each point of the plane represents a definite optical depth τ and a definite individual velocity V .

interval from τ_a to τ_b (13a), as indicated in the figure by the two dotted horizontal lines.

Now an atom at rest can only scatter or emit after recombination radiation of the natural frequency ν_a , or very close to it. For moving atoms, the frequency of interaction for scattering and production (by recombination) is different for the outward beam I_ν and the inward beam I_ν' , thus

$$\nu \text{ (for } I_\nu) = \nu_a \left(1 + \frac{V}{c} \right), \quad \nu \text{ (for } I_\nu') = \nu_a \left(1 - \frac{V}{c} \right) \quad (14)$$

on account of the Doppler effect. Each class $V \pm \frac{1}{2}dV$ with the outward and inward radiation of the frequencies corresponding to (14) acts by itself and does not interact with any other class.

Now it is inconvenient to have different frequencies for I and I' . In dealing with the group of atoms of class $V \pm \frac{1}{2}dV$ we use therefore a *system of reference moving with velocity V* . In this " V -system," which is the same for all parts of the nebula, the atomic velocities are homogeneously distributed within the interval dV (for the part of τ from τ_a to τ_b , in which they

occur), provided that we take dV small enough.* The frequency in the V -system, which will be denoted by ν' , is within the interval $d\nu'$ given by

$$\frac{1}{2}d\nu' > \nu' - \nu_a > -\frac{1}{2}d\nu', \quad \frac{d\nu'}{\Delta} = \frac{dV}{2w}, \quad (15)$$

homogeneously distributed within this interval (for $\tau_a < \tau < \tau_b$).

The frequency interval (15) is no longer different for I and I' .† Hence we may introduce the *radiation intensities* $I_{\nu'}d\nu'$, $I'_{\nu'}d\nu'$ in quanta per sec. per unit solid angle, and the net flux $\pi F_{\nu'}d\nu'$.

The number of normal H -atoms per cm.³ within dV is

$$C_a \frac{dV}{2w} \quad \text{for} \quad \tau_a < \tau < \tau_b, \quad (16)$$

and zero outside this interval. But the number of atoms (16) in the velocity interval dV corresponds to the number C_a within $2w$, hence giving the absorption coefficient α_l (6) for the thermal width $2w$:

$$\alpha_{\nu'} = \alpha_l \quad \text{for} \quad \tau_a < \tau < \tau_b, \quad (17)$$

and zero outside this interval.

The optical depth for the line L_a in the V -system is

$$\sigma_{\nu'} = \int_0^{\xi} \alpha_{\nu'} C_a d\xi. \quad (18)$$

From (17) and (18) we obtain

$$\begin{array}{lll} \sigma_{\nu'} = 0, & \sigma_{\nu'} = \sigma - \sigma_a, & \sigma_{\nu'} = \sigma_b - \sigma_a, \\ \tau < \tau_a, & \tau_a < \tau < \tau_b, & \tau > \tau_b. \end{array} \quad (19)$$

Summing Up.—Atoms of group $V \pm \frac{1}{2}dV$, interacting frequencies $\nu_a \pm \frac{1}{2}d\nu'$ (15), only occur from τ_a to τ_b (13a) (or fig. 1), with a line absorption coefficient α_l (6), and this group may be treated by itself, the system of reference being that moving with a velocity V .

We shall now consider the equations of radiative transfer for L_a . The original flux of L_c from the star is πS_c , absorbed with coefficient α_c (optical depth τ), each quantum considered to be subsequently transformed into one quantum L_a (section 2).

Hence the number of L_a -quanta produced per sec. per cm.² within $d\tau$ is

$$4\pi E d\tau = \pi S_c e^{-\tau} d\tau = \pi S_c u e^{-u\sigma} d\sigma. \quad (20)$$

These quanta are considered to be produced uniformly within the

* However, we should remark that for the following treatment the interval dV should be large compared with the radiation damping width corresponding to $\tau_b - \tau_a$, otherwise atoms of different classes are no more independent.

† Had we considered the velocity interval V to $V + dV$, the frequency interval for I in the V -system would be just adjoining the frequency interval for I' , but not coinciding with it. This is the reason why we used $\pm \frac{1}{2}dV$.

total width $2w$, corresponding to the frequency width Δ . Hence, within dv' (15)

$$4\pi E_{\nu'} d\tau dv' = \pi S_c \frac{u}{\Delta} e^{-u\sigma} d\sigma dv', \quad (21)$$

within $\tau_a < \tau < \tau_b$, since atoms interacting with the frequencies specified only occur within this interval.

Let the total number of quanta emitted per second within $d\sigma dv'$ be

$$4\pi B_{\nu'} d\sigma dv'. \quad (22)$$

The *equations of radiative transfer for L_a* are then, using the Schwarzschild-Schuster approximation,*

$$-\frac{1}{2} \frac{dI_{\nu'}}{d\sigma} = I_{\nu'} - B_{\nu'}, \quad (23)$$

$$-\frac{1}{2} \frac{dI'_{\nu'}}{d\sigma} = B_{\nu'} - I'_{\nu'}, \quad (24)$$

within $\tau_a < \tau < \tau_b$. Our definitions are the same as customary, except that we have measured the optical depth *outward*, which is taken into account by the minus sign on the left-hand side of each equation.

The number of quanta scattered is the number emitted minus the number created—that is, $4\pi B_{\nu'} d\sigma dv' - 4\pi E_{\nu'} d\tau dv'$. Putting this equal to the number absorbed, $2\pi(I_{\nu'} + I'_{\nu'}) d\sigma dv'$, we obtain the *condition of radiative equilibrium*:

$$2\left(B_{\nu'} - \frac{S_c}{4} \frac{u}{\Delta} e^{-u\sigma}\right) = I_{\nu'} + I'_{\nu'}, \quad (25)$$

where (21) has been substituted.

The *net outward flux* $\pi F_{\nu'}$ is given by

$$F_{\nu'} = I_{\nu'} - I'_{\nu'}. \quad (26)$$

Substituting (25) in (23) and (24), we obtain

$$\frac{dI_{\nu'}}{d\sigma} = -F_{\nu'} + \frac{S_c}{2} \frac{u}{\Delta} e^{-u\sigma}, \quad (27)$$

$$\frac{dI'_{\nu'}}{d\sigma} = -F_{\nu'} - \frac{S_c}{2} \frac{u}{\Delta} e^{-u\sigma}. \quad (28)$$

Subtracting (28) from (27), using (26),

$$\frac{dF_{\nu'}}{d\sigma} = S_c \frac{u}{\Delta} e^{-u\sigma}; \quad (29)$$

or, integrated,

$$F_{\nu'} = I_{\nu'} - I'_{\nu'} = P - \frac{S_c}{\Delta} e^{-u\sigma}. \quad (30)$$

* Ambarzumian's quantities $\tau_1 - \tau$, $t_1 - t$, K , C , $\frac{I}{\omega}$ are our quantities τ , σ , I , B , u .

That he uses I and B for the L_c , and we use it for L_a does not need to give rise to confusion, the more so since we have added the subscript ν' .

Substituting (30) in (27),

$$\frac{dI_{\nu'}}{d\sigma} = -P + \frac{S_c}{\Delta} e^{-u\sigma} + \frac{S_c}{2} \frac{u}{\Delta} e^{-u\sigma}, \quad (31)$$

and integrating, we obtain

$$I_{\nu'} = Q - P\sigma - \frac{S_c}{\Delta} \frac{1}{u} e^{-u\sigma} - \frac{1}{2} \frac{S_c}{\Delta} e^{-u\sigma}, \quad (32)$$

$$I'_{\nu'} = Q - P(1 + \sigma) - \frac{S_c}{\Delta} \frac{1}{u} e^{-u\sigma} + \frac{1}{2} \frac{S_c}{\Delta} e^{-u\sigma}, \quad \tau_a < \tau < \tau_b. \quad (33)$$

Equation (33) follows from (32) by subtracting (30).

Introduction of the boundary conditions, $I_{\nu'} = 0$ at $\tau = \tau_a$ and $I'_{\nu'} = 0$ at $\tau = \tau_b$, gives us the desired expressions for P and Q to be substituted in (30), (32) and (33):

$$\left. \begin{aligned} P &= \frac{1}{1 + \sigma_b - \sigma_a} \frac{S_c}{\Delta} \left[\frac{1}{u} (e^{-u\sigma_a} - e^{-u\sigma_b}) + \frac{1}{2} (e^{-u\sigma_a} + e^{-u\sigma_b}) \right] \\ Q &= \frac{S_c}{\Delta} \left(\frac{1}{u} + \frac{1}{2} \right) e^{-u\sigma_a} + P\sigma_a \end{aligned} \right\}. \quad (34)$$

Equations (32), (33) and (34) give $I_{\nu'}$ and $I'_{\nu'}$ for class $V \pm \frac{1}{2} dV$. The intensities and corresponding frequencies may, if desired, be easily reduced to the rest system, using (14), so that the problem of the distribution of line radiation due to L_α among the different frequencies for outgoing and ingoing flux is formally solved.

For $\tau < \tau_a$, $I'_{\nu'}$ is constant and equal to its value at τ_a ; similarly, for $\tau > \tau_b$, $I_{\nu'}$ is constant.

For our purpose the *net flux* is most important. Equations (30) and (34) give

$$F_{\nu'} = -\frac{S_c}{\Delta} e^{-u\sigma} + \frac{1}{1 + \sigma_b - \sigma_a} \frac{S_c}{\Delta} \left[\frac{1}{u} (e^{-u\sigma_a} - e^{-u\sigma_b}) + \frac{1}{2} (e^{-u\sigma_a} + e^{-u\sigma_b}) \right], \quad (35)$$

where (13a) should be substituted.

6. *Radiation-pressure in an Expanding Nebula, v Proportional to τ .*—The force due to radiation-pressure L_α on atoms in the velocity interval dV , within $d\sigma$, per cm^2 is

$$\pi F_{\nu'} dv' h\nu' \frac{d\sigma}{c}, \quad \tau_a < \tau < \tau_b,$$

since it is the net energy flux per cm^2 multiplied by $\frac{d\sigma_{\nu'}}{c}$,* and $d\sigma_{\nu'} = d\sigma$ (19) for the interval of τ specified.

We first substitute $\frac{\Delta}{2w} dV$ for dv' , using (15).

At a certain assigned optical depth $\bar{\tau}$, corresponding to the mechanical velocity $\bar{v}$ by (10), the force per cm^2 on an element $d\sigma$ is then obtained by

* E. A. Milne, *Handbuch der Astrophysik*, 3, chap. ii, p. 65. The result holds also if, as in the present case, energy is produced within the element $d\tau$, since this energy is emitted evenly in all directions.

integrating between the limits $V = \bar{v} - w$ and $V = \bar{v} + w$ (the vertical strip parallel to AB in fig. 2), hence

$$dP_r = \frac{h\nu_\alpha}{c} d\sigma \cdot \pi F_{\text{eff}}, \quad (36)$$

$$F_{\text{eff}} = \int_{V=\bar{v}-w}^{V=\bar{v}+w} \frac{\Delta}{2w} F_{\nu'} dV = \int_{W=-w}^{W=+w} \frac{\Delta}{2w} F_{\nu'} dW, \quad (36a)$$

where we have reintroduced the thermal velocity W (11), and put $\nu_\alpha = \nu'$, which is rigorously true, since dV is infinitesimal.

The quantity πF_{eff} in (36) represents the flux of quanta per cm^2 per sec. which can interact with atoms at the point $\bar{\tau}$ considered, and are therefore effective. In a non-expanding nebula all quanta are effective. That is why (36) for an expanding nebula reduces to (I) for a stationary nebula, if $F_{\text{eff}} = F$. We make use of this, because Table I contains the acceleration due to radiation-pressure, if $F = S_c$ in (I). Hence *Table I also contains the acceleration due to radiation-pressure of L_α , if $F_{\text{eff}} = S_c$ in (36).*

If we know τ as function of v , that is the function ϕ in (10), substitute τ_a and τ_b (13a) in (35) for $F_{\nu'}$, and then are able to evaluate the integral, the problem of finding the force of radiation-pressure is solved.

We shall solve the simple case in which v is proportional to τ :

$$v = \frac{\tau}{a} + v_0, \quad (37)$$

or

$$\tau = \phi(v) = \bar{\tau} + a(v - \bar{v}), \quad (38)$$

where $\bar{\tau}$, $\bar{v}$ represent the optical depth and mechanical velocity at the point where we wish to determine the radiation-pressure.

Equations (13a) and (38) give, using (11),

$$\left. \begin{aligned} \tau_a &= \bar{\tau} + aW - aw \\ \tau_b &= \bar{\tau} + aW + aw \end{aligned} \right\} \quad (39)$$

The expression (35) for $F_{\nu'}$ holds for $\tau_a < \tau < \tau_b$ (13), which is equivalent to (6a) $w > W > -w$, but these are precisely the limits of integration of (36a). Substituting (35) in (36a), and putting $u\sigma = \tau$ (8), we have

$$F_{\text{eff}} = \frac{S_c}{2w} \int_{-w}^{+w} \left[-e^{-\tau} + \frac{(e^{-\tau_a} - e^{-\tau_b}) + \frac{1}{2}u(e^{-\tau_a} + e^{-\tau_b})}{u - \tau_b - \tau_a} \right] dW. \quad (40)$$

Substitution of (39) and integration gives

$$F_{\text{eff}} = S_c e^{-\bar{\tau}} \left[-1 + \frac{(e^{+aw} - e^{-aw})^2 + \frac{1}{2}u(e^{+2aw} - e^{-2aw})}{2aw(u + 2aw)} \right]. \quad (41)$$

The limits of integration (13a) are applicable only for those parts of the nebula where $\tau_a \geq 0$ and $\tau_b \leq \tau_1$, the total optical depth (see fig. 2). Near the inner boundary, where (13a) gives $\tau_a < 0$, one should use the lower limit $\tau_a = 0$. Near the outer boundary, where (13a) gives $\tau_b > \tau_1$, one should use the upper limit $\tau_b = \tau_1$. Hence equation (41) holds *with the exclusion of the inner edge τ_a (13a) < 0 and of the outer edge τ_b (13a) $> \tau_1$.*

For the discussion of (41) we introduce *the effective range* v_L of the velocity of luminous matter, that is the range $0 < \tau < 1$, within which an appreciable fraction of the luminosity (equal to $1 - \frac{1}{e} = 0.63$) is concentrated, and which causes the observed Doppler broadening of the line due to mass motion, if the thermal width is small. It is given by

$$v_L = v - v_0 \quad \text{for} \quad \tau = 1. \quad (42)$$

This, with (37) and $\tau = 1$, gives

$$v_L = \frac{l}{a}, \quad aw = \frac{w}{v_L}. \quad (43)$$

Put $u = lw$, write $aw = \frac{w}{v_L}$ (43), and expand (41) into powers of $\frac{w}{v_L}$ and $\frac{l}{a}$ and products of these powers. The result is the *approximate expression*

$$F_{\text{eff}} = S_e e^{-\tau} \left[\frac{1}{3} \left(\frac{w}{v_L} \right)^2 \right]. \quad (44)$$

The relative error of this approximation for the factor in square brackets is the larger of the two quantities $\left(\frac{w}{v_L} \right)^4$ and $\frac{l}{a} \left(\frac{w}{v_L} \right)^2$. The quantity $\frac{l}{a}$ is always smaller than 0.81×10^{-3} , since by (7) $l = 0.81 \times 10^{-5}$ and v_L or $\frac{l}{a}$ practically never exceeds 100 km./sec. Since the right-hand side of (44) is positive and v is considered positive when directed towards the right in fig. 1, the flux F_{eff} is also directed away from the exciting star.

Since we have seen above that Table I contains the acceleration a_r for $F_{\text{eff}} = S_e$, we obtain from (44) and Table I :

$$a_r = e^{-\tau} \frac{1}{3} \frac{w}{v_L} \frac{4.56}{v_L} \text{ km./sec. per year, for } T = 40,000^\circ, \quad i = 260, \quad (45)$$

for the acceleration due to L_α in a planetary nebula of radius 10,000 astronomical units. It is directed away from the exciting star. For such a nebula an observed luminous width of about 20 km./sec. may be adopted, on the average (section 3). We have, for a stationary nebula, $w = 10$ km./sec., hence $a_r \sim 456$ km./sec. in 1000 years, for $T = 40,000^\circ$ (Table I). On the other hand, if the nebula is expanding, and we therefore interpret the luminous width by $v_L = 10$ km./sec., (45) gives us $a_r \sim 15$ km./sec. in 1000 years, if $w = 1$ km./sec., and $a_r \sim 1.5$ km./sec. in 1000 years, if $w = 0.1$ km./sec.*

The result expressed in (44), (45) means, therefore, *that the radiation-pressure due to L_α may be reduced very considerably by the expansion of the*

* Dividing the radius 10,000 astronomical units by the velocity 10 km./sec., we obtain about 5000 years for the life of a planetary, provided that a_r is small. In this time radiation-pressure would have set up a velocity of the order 7.5 km./sec. in the second case considered, $w = 0.1$ km./sec. The line would remain narrow and the discrepancy with observation discussed in section 3 be removed.

nebula, provided that the ratio of thermal to luminous half-width w/v_L is small. It even rigorously approaches zero, as w/v_L approaches zero, since then the terms left out in the expansion are all infinitely small as compared with the term retained.

7. *The Friction Force of Radiation and the "Thermal" Width of Neutral Atoms.*—In this section we wish to call attention to one cause which tends to make w/v_L small. Our considerations are only concerned with orders of magnitude. Again we take a stellar temperature of $40,000^\circ$ and a nebular radius of 10,000 astronomical units as the typical example.

The electrons ϵ in the nebula have a certain mean kinetic energy, which may formally be represented as a temperature, the electron temperature T_ϵ . For the protons π , which are continually colliding with the electrons, we shall adopt the same mean kinetic energy, as an upper limit at any rate. Then, the thermal velocity w_ϵ of electrons and w_π of protons is given by

$$w_\epsilon = 4.57 \sqrt{T_\epsilon} \text{ km./sec.}, \quad w_\pi = 0.107 \sqrt{T_\epsilon} \text{ km./sec.}, \quad (46)$$

obtained from the mean square velocity of the kinetic theory half-width.

We substitute (39) in the integrand of (40) and expand into powers with the same approximations as for (44), remembering that W is of the same order as w . The result is

$$dF_{\text{eff}} = S_c e^{-\tau} \left\{ -\frac{W}{v_L} + \frac{1}{2} \left(\frac{W}{v_L} \right)^2 - \frac{1}{6} \left(\frac{W}{v_L} \right)^3 + \frac{1}{6} \left(\frac{w}{v_L} \right)^2 \left(1 - \frac{W}{v_L} \right) \right\} \frac{dW}{2w}, \quad (47)$$

the relative error being the larger of the two quantities $\left(\frac{w}{v_L} \right)^4$ and $\frac{l}{a} \cdot \left(\frac{w}{v_L} \right)^2$.

Retaining only the main term, we have :

$$dF_{\text{eff}} = -S_c e^{-\tau} \frac{W}{v_L} \frac{dW}{2w}, \quad +w > W > -w; \quad (48)$$

representing the effective flux corresponding to the velocity interval dW for a volume element at an optical depth τ .

For $|W| > w$ the atoms are interacting with radiation due to a strip above B or below A in fig. 2. Then the flux has the value of the emerging radiation for this strip; and is approximately given by

$$dF_{\text{eff}} = -S_c e^{-\tau} \frac{w}{v_L} \frac{dW}{2w}, \quad |W| > w. \quad (49)$$

Now $dF_{\text{eff}} = S_c e^{-\tau} \frac{dW}{2w}$ or $F_{\text{eff}} = S_c e^{-\tau} \sim S_c$ gives the acceleration (II),

if acting on protons plus neutral atoms, hence $i \times$ (II) during the time it acts on a neutral atom. Since $i = 260$ (Table I), the acceleration A_r of a neutral atom becomes, according to (II), (48) and (49),

$$A_r \sim -\frac{1.185 \times 10^3}{v_L w} W \text{ km./sec. per year}, \quad +w > W > -w. \quad (50)$$

$$A_r \sim -\frac{1.185 \times 10^3}{v_L} \text{ km./sec. per year}, \quad |W| > w. \quad (51)$$

Equation (50) represents a friction proportional to the velocity W as long as $|W| < w$, and equation (51) a constant friction, when $|W| > w$.

Now the life of a neutral atom—that is, the time it stays in the neutral state between a recombination and subsequent ionisation—is given by

$$\text{Life} \sim \frac{1}{\pi S_c \alpha_c} = 1.063 \times 10^7 \text{ sec.} = 0.337 \text{ year}, \quad (52)$$

where $\pi S_c = 1.747 \times 10^{10}$ quanta per cm.² per sec. has been substituted for the typical nebula.

The electron temperature T_e certainly is much lower than the stellar temperature of 40,000°, since the electrons lose much of their energy in exciting the nebulum lines.

For $T_e = 10,000^\circ$, we have (46) $w_\pi = 10.7$ km./sec. This is the order of the velocity W for the neutral atom just after recombination. Then the neutral atom is decelerated. As long as $|W| > w$, the loss of velocity is given by (51), that is (assuming $v_L = 10$ km./sec.) 118.5 km./sec. in 1 year, or 40 km./sec. during its life of 0.337 year. Hence after 1/4 of its neutral life the atom would have lost its velocity due to this cause.

Similarly, for $T_e = 2500^\circ$, $w_\pi = 5.4$ km./sec., and the atom would have lost its velocity in 1/8 of its neutral life.

Though the above considerations are by no means rigorous, and many other causes may be operative, they tend to show that w/v_L may be small, even when the initial velocity on recombination is comparable with v_L .

We remark that the cause discussed here would not show in the observed widths of the emission lines, since on recombination the atom still has the full ion velocity w_π .

8. *An Alternative. Line Width in the Orion Nebula.*—For most of the atoms and ions occurring in nebulae, e.g. He , O^+ , O^{++} , etc., the second level is metastable. If an atom or ion emits its spectrum, the resonance lines, by a process similar to that occurring in the Lyman series of H (section 2), will all be reduced to the first line of the series. But this line originates from a metastable level. This means that the probability of emission is low, and the same may be said for the probability of absorption, because of Einstein's relation between the A and the B. This is confirmed by observation, since none of the nebulum resonance lines has ever been observed in absorption in spectra of stars surrounded by nebulosity. Because most atoms and ions are in the lowest state, and the one resonance line leaving the nebula has an absorption coefficient practically zero, the radiation-pressure due to line spectra of these atoms and ions is negligible.

It is quite different with H and the analogous spectrum of He^+ . It is true that a fraction of the lowest of the two sublevels of H is metastable, but exactly coinciding with this metastable level is a stable level, so that the average absorption coefficient for the corresponding line is not small and the reasoning given above for the other spectra does not apply. As we have seen in the previous sections, the radiation-pressure due to L_α is very considerable in a stationary nebula, and we had to look for reasons why it is much lower than this, in order to avoid a conflict with observation.

Now one might suggest that the one sublevel of H is metastable in the true sense, so that the H -atom can stay there for days and only lose its energy by a collision of the second kind with an electron (the time between electron collisions is of the order one hour). But then this electron will have to get rid of its energy and, especially for nebulae with low temperature stars emitting pure line spectra, there seems no other possibility but that its energy is given up in exciting the nebulum. For every Balmer quantum emitted, the energy of one L_{α} -quantum would then be transformed into nebulum lines, and they would always be very strong. Now this is certainly not the case. For nebulae of low excitation it may even be that the N_1 and N_2 lines are too faint to be observed, whereas H_{β} is clearly recorded (*e.g.* the case of I.C. 5146 observed by Hubble and cited in *Ap. J.*, 65, 68, 1927).

There seems hardly any alternative but that offered in the present paper : that the nebula is expanding and the thermal width small compared with the observed luminous width.

The observations by Baade, Goos, Koch and Minkowski * of the line widths in the Orion nebula would at first sight seem very unfavourable to the assumption of a small thermal width.† Under certain assumptions these authors find for hydrogen a mean thermal velocity (our w , if we leave out the cause of section 7) of about 17.6 km./sec. and a mean mechanical velocity (our v) of about 10 km./sec., assuming for the latter an irregular distribution, so that it may equally well be positive or negative.‡

The main facts to be explained are the ratios of the line widths Δ for O_{III} , O_{II} and H :

$$\frac{\Delta O_{III}}{\Delta H} = \frac{1}{\sqrt{3.05}} = \frac{1}{1.75} \quad \frac{\Delta O_{II}}{\Delta H} = \frac{1}{\sqrt{2.75}} = \frac{1}{1.66}, \quad (59)$$

the former result being due to Baade and his co-workers, the latter to Buisson, Fabry and Bourget. Since this ratio is smaller than 1 but larger than $1/\sqrt{16}$, it was concluded that the motion is neither purely mechanical nor purely thermal, but that thermal and mechanical velocities are comparable.

Now it seems certain that the luminous portions of the nebula due to H , O_{III} and O_{II} have not the same extension in space. Reynolds § found that the O_{II} picture has a much larger extent than O_{III} , which may be theoretically understood, since the portions of the nebular surface farther away from the star have less ionisation. Similarly one may expect more O_{III} for the superficial and more O_{II} for the inner layers, the latter being farthest away from the exciting star, if one applies to the Orion nebula a consideration similar to that by Bowen for planetaries. Keeping this in

* W. Baade, F. Goos, P. P. Koch and R. Minkowski, *Zeitschr. f. Astrophys.*, 6, 355, 1933.

† Apart from the cause mentioned in the previous section, which can never show up in emission lines.

‡ These values are the averages from their figures 12 and 13, multiplied by a factor 2, since the right-hand sides of their formulæ (9) and (10) are too low by this factor.

§ Reynolds, *M.N.*, 79, 561, 1919.

mind, we may give an interpretation of the line widths (53) solely on the basis of mechanical motions.

We assume the exciting star to be outside the nebula—that is, between us and the nebula. The lower part of fig. 1 represents a fraction of the central part of the nebula; the exciting stars are thought towards the left. In the outer layers one has predominantly O_{III} , but going inward the ionisation decreases, and finally O_{II} gets predominant. For the sake of definiteness we assume that the outer half of the luminous part is O_{III} and the inner half O_{II} . But O_{III} as well as O_{II} are mainly excited by the photo-electrons from H , hence the H luminosity covers both the O_{III} and O_{II} regions. We may consider the right-hand boundary of O_{II} and H at rest. Then, if we tentatively assume an expansion of the type v proportional to τ , the outer (viz. left-hand) boundary of O_{II} moves with half the velocity of the outer boundary of O_{III} and H . The result of these purely mechanical motions is that the width of O_{III} is equal to that of O_{II} and only one-half of that of H . But, since O_{III} and O_{II} are partly overlapping, one may expect the ratios $\Delta O_{III}/\Delta H$ and $\Delta O_{II}/\Delta H$ to be somewhat larger than $1/2$, which is in agreement with the observed values (53).

It should be noted that this purely mechanical interpretation requires that the centre of the O_{III} lines N_1 and N_2 is displaced towards the violet, and the centre of the O_{II} line 3727 towards the red with respect to H . At present it is not possible to verify this prediction, since the theoretical wavelengths of O_{III} and O_{II} are not known with sufficient accuracy, and the values adopted for N_1 and N_2 , e.g.,* are to an appreciable extent based on the Orion nebula itself, and otherwise on two nebulae with rather fuzzy lines.

In other respects, however, a verification is possible. At the outer edge of the nebula it may be that the material is dissipated, as might be concluded from section 6. This means that the violet edge of the O_{III} and H lines may become fuzzy. This was indeed observed by Campbell and Moore† for a certain portion of the central region of the Orion nebula. It would be very interesting to observe the O_{II} line 3727 in the same region, since, on our interpretation, we would expect the violet edge of this line to be sharp.

With increasing distance from the trapezium, the depth of the O_{III} region should decrease, because of the greater dilution of stellar radiation. Hence the ratio $\Delta O_{III}/\Delta H$ should decrease. This is indeed the case, as follows from figs. 10 and 11 of the paper of Baade and his co-workers. We would also expect the ratio $\Delta O_{II}/\Delta H$ to increase farther away from the trapezium, which might be a subject for future observation.

Summary.—Ambarzumian has calculated the radiation-pressure due to the first line of the Lyman series of hydrogen in a stationary nebula. If it remained of the same order in an expanding nebula, it would produce a velocity of at least 200 km./sec. in the envelope of a planetary nebula of radius 10,000 astronomical units. This is contrary to the facts, since the observed line width requires an average velocity of about 10 km./sec. for

* Campbell and Moore, *Lick Obs. Bull.*, 9, 91, 1915.

† Campbell and Moore, *Publ. Lick Obs.*, 13, 99, 1918.

planetary nebulae of radii up to 10,000 astronomical units. It is shown, by an extension of Ambarzumian's methods, that for an expanding nebula, where the mechanical velocity v is proportional to the optical depth, the radiation-pressure is only $\frac{1}{3} \left(\frac{w}{v_L} \right)^2$ of what it would be for a stationary nebula with the same thermal width w , if v_L is the velocity range of the main luminous portion of the envelope. For a stellar temperature of $40,000^\circ$ and a nebular radius of 10,000 astronomical units, the order of the acceleration due to radiation-pressure is given by $\frac{1}{3} \frac{w}{v_L} \frac{4.56}{v_L}$ in km./sec. per year. Hence a reduction of radiation-pressure as required by observation may be obtained, if the thermal width w is much smaller than the luminous width v_L . One reason for assuming a rather small thermal width is given. It is also shown that the observations of Baade, Goos, Koch and Minkowski for the line width of the Orion nebula may be interpreted on the assumption of a very small thermal width, so that the width might practically be due to mechanical causes only.

The work presented in this paper was carried out at the Observatory of Oxford during the summer of 1934. To Professor Milne I am indebted for discussions on general aspects of the problem of radiation-pressure. I wish to express my sincere thanks to Professor H. H. Plaskett for the active interest taken by him in the development of all problems dealt with. The last section is largely the result of discussions with him.

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OSCILLATOR STRENGTHS AND THE CONTINUOUS ABSORPTION COEFFICIENT.

R. v. d. R. Woolley, Ph.D.

§ 1. An account of the nature and properties of certain numbers connected with the intensities of lines in spectra, called *oscillator strengths*, will be found in an article by Weisskopf in *The Observatory*.* We may recall here that in the classical electromagnetic theory an electron which executes a simple harmonic motion $x = A \cos(2\pi\nu_0 t + \alpha)$ will absorb radiation from a field of radiation density ρ_ν at the rate

$$\rho_\nu \frac{\gamma e^2/m}{(\nu - \nu_0)^2 + \gamma^2}, \quad \text{I. I}$$

* *Observatory*, 56, 291, 1933.

where $\gamma = \omega/4\pi$ and $\omega = \frac{8\pi^2 e^2 \nu_0^2}{3mc^3}$ is the "classical damping constant." This absorption has a pronounced maximum at $\nu = \nu_0$, since γ is small, and is, in fact, line absorption.

The mass coefficient for continuous radiation is defined as follows. Consider a cylinder of unit cross-section and of length ds . Let the flow of radiation per second between the frequencies ν and $\nu + d\nu$ be $\mathfrak{J}_\nu d\nu$. Let the flow $\mathfrak{J}_\nu$ lose by absorption in the cylinder the amount $\mathfrak{J}_\nu k_\nu \rho ds$, where ρ is the mass density of the atmosphere. Then k_ν is the mass coefficient of absorption.

In the case of selective absorption by a particular atom we may denote the mass-absorption coefficient by σ_ν : then if there are N atoms each of mass M per c.c. of the atmosphere, the amount of radiation selectively absorbed per second in the cylinder is $\mathfrak{J}_\nu \sigma_\nu N M ds$. But $\mathfrak{J}_\nu = c \times \rho_\nu$, where ρ_ν is the radiation density. Then from (I.1)

$$\sigma_\nu = \frac{1}{M} \frac{e^2}{mc} \frac{\gamma}{(\nu - \nu_0)^2 + \gamma^2}.$$

The *atomic* coefficient of line absorption is $s_\nu = M\sigma_\nu$ or

$$s_\nu = \frac{e^2}{mc} \frac{\gamma}{(\nu - \nu_0)^2 + \gamma^2}.$$

This expression is replaced on the quantum theory by its analogue

$$s_\nu = \frac{e^2}{mc} \times \frac{f\delta}{(\nu - \nu_0)^2 + \delta^2},$$

where δ is not necessarily equal to γ and f is a numerical factor called the *oscillator strength* of the line. (Weisskopf uses a subscript notation to distinguish between upward and downward transitions. Our f is his f_{ki} where $E_k > E_i$.)

The total amount of radiation scattered per sec. by the classical bound electron is $\pi e^2/m \times \rho_\nu$, and the amount scattered per sec. by the quantum electron is $\pi e^2/m \times f\rho_\nu$, so that in this respect the quantum electron is equivalent to f classical oscillators (whence the name oscillator strength for f). The quantities f are related in a simple way with the Einstein transition probabilities of the lines.

We have in Weisskopf's notation

$$f_{ki} = \frac{m}{\pi e^2} \times h\nu_{ki} \times B_{ki}.$$

Or, using the notation given by Eddington in *I.C.S.* and dropping the suffixes from the f 's,

$$f = \frac{m}{\pi e^2} \times h\nu_0 \times a_{12}.$$

The quantities δ do not appear in the subsequent argument, but it may

be as well to state, for completeness, how they are calculated.* Let the line be formed as the result of transitions between state 2 and state 1. Then $4\pi\delta$ is the sum of the spontaneous transitions from state 2 as well as from state 1, or

$$4\pi\delta = \sum_x b_{2x} + \sum_x b_{1x}.$$

(In an earlier notation used by the present writer,† the quantity $\frac{8\pi^2 e^2 \nu_0^2}{3mc^3}$ was denoted by ω_{kl} (the suffix meaning "klassisch"), and $\frac{\omega}{4\pi}$ was used for δ .)

In the special case of the resonance line (from upper state 2 to lower state 1) there is only one transition and $4\pi\delta = b_{21}$. Comparing δ with the classical half-width γ we find after a little reduction

$$\delta = 3 \frac{g_1}{g_2} f \cdot \gamma.$$

To go on to a property of the oscillator strengths f directly connected with the continuous absorption coefficient, the oscillators are conceived as being divided amongst the lines in a series and also the continuous spectrum at the head of the series. The *Sum Rule* then states that the sum of the f 's connected with any given state (negative in the case of spontaneous transitions) is equal to the valency of the atom, or in our notation

$$\sum_{\text{abs}} f - \sum_{\text{em}} \frac{g_x}{g_2} f = s,$$

x being any state to which there is a spontaneous transition from the given state 2. This summation including oscillators in the continuous spectrum, we can calculate the oscillator strength of the continuous spectrum by subtracting the sum of the line oscillator strengths from the valency s . We can go a step further and calculate the actual coefficient of absorption.

To take a concrete example, consider the simplest case, the Lyman series in the hydrogen spectrum. These lines are transitions of type $1S - nP$, and there are no other transitions from the $1S$ state, which is the ground state of hydrogen. We can add up the f values for these lines. If there were no continuous absorption beyond the head of the series at 912 Å., the sum of the line f 's would be unity, since $s = 1$ for hydrogen; but in fact the sum of the line f 's is 0.564, leaving 0.436 as the sum of the oscillator strengths of the transitions forming a continuous band beyond (*i.e.* on the short-wave side of) 912 Å.

Let the coefficient of absorption of continuous radiation at the head of a series be k_ν at frequency ν , and let $\phi_\nu d\nu$ be the number of oscillators between ν and $\nu + d\nu$, the oscillators representing continuous absorption being spread out continuously in frequency according to a law which we have yet to determine.

Let ν_0 be the frequency of the series limit and let f_c be the sum of the

* See Weisskopf, *op. cit.*

† *M.N.*, 93, 691, 1933.

oscillator strengths of the continuous absorption as given by the sum rule. Then

$$\int_{\nu_0}^{\infty} \phi_{\nu} d\nu = f_c.$$

Consider a frequency ν_2 as affected by a unit oscillator whose central frequency is ν_1 . The coefficient of absorption at ν_2 due to this oscillator is

$$\frac{e^2}{mc} \frac{\delta}{(\nu_2 - \nu_1)^2 + \delta^2}.$$

The absorption at ν_2 due to $\phi_{\nu_1} d\nu$ oscillators between ν_1 and $\nu_1 + d\nu$ is

$$\frac{e^2}{mc} \frac{\phi_{\nu_1} d\nu \times \delta}{(\nu_2 - \nu_1)^2 + \delta^2}$$

and the total absorption at ν_2 is

$$\frac{e^2}{mc} \int_0^{\infty} \frac{\delta \cdot \phi_{\nu} d\nu}{(\nu_2 - \nu)^2 + \delta^2}.$$

Now the effective range of the integral is very small, being a range comparable with δ , the half-width of a line. As ϕ_{ν} varies slowly with ν we may take it as being constant over a short range of frequency, and we get

$$\frac{\pi e^2}{mc} \times \phi_{\nu_2}$$

for the coefficient of absorption at ν_2 .

We now suppose that the absorption at the head of a series follows the law $k \propto \lambda^3$. (To establish this we may either appeal to experiment, as far as it goes—that is to say, the law is known to hold in the X-ray region, although some experiments in the optical region do not confirm it closely—or we may appeal to Kramers' theory. At any rate, we must make this assumption for the sake of definiteness; but the application of the oscillator strengths to the calculation of k could proceed equally well with the general law $k \propto \lambda^n$.)

Let $k_{\nu} = \alpha \lambda^3$. Then

$$\frac{\pi e^2}{mc} \phi_{\nu} = \alpha \lambda^3 \quad (\text{for all } \lambda < \lambda_0),$$

and since $f_c = \int_{\nu_0}^{\infty} \phi_{\nu} d\nu$ we must have $f_c = \alpha m c^4 / 2 \pi e^2 \nu_0^2$

and

$$k_{\nu} = \alpha \lambda^3 = \frac{2 \pi e^2 \nu_0^2 f_c}{m c^4} \times \lambda^3.$$

Notice that

$$\phi_{\nu} = \frac{2 \nu_0^2 f_c \lambda^3}{c^3}.$$

(The corresponding results for $k_{\nu} = \alpha \lambda^n$ are $k_{\nu} = \frac{(n-1) \pi e^2 \nu_0^{n-1} f_c}{m c^{n+1}} \lambda^n$ and $\phi_{\nu} = (n-1) \nu_0^{n-1} f_c \lambda^n / c^n$.)

k_ν is the continuous absorption coefficient per atom in the state considered. If this state is not the ground state, but is a state whose excitation potential is $h\nu_s$ volts and statistical weight g_s , we shall have for the coefficient of absorption per atom

$$k_\nu = \lambda^3 \times \frac{2\pi e^2 \nu_0^2 f_c}{mc^4} \times \frac{g_s}{g_0} \exp\left(-\frac{h\nu_s}{RT}\right).$$

We can now make a direct comparison of this result with Kramers' theory, in the case of hydrogen. In the form given by Strömgren,* Biermann † and Unsöld ‡ we have the following expression for the absorption coefficient at the head of the n th series, per electron in the n th state:—

$$k(n, \nu) = \frac{64\pi^4}{3\sqrt{3}} \frac{me^{10}}{c^4 h^6 n^5} \times g(n, \nu) \times \lambda^3,$$

where $g(n, \nu)$ is a numerical factor nearly equal to unity.

The number of electrons in the n th state, per electron in the ground state, is $n^2 \exp\left(-\frac{h\nu_n}{RT}\right)$. Let $\bar{g}$ be the mean value of $g(n, \nu)$.

Comparing the two expressions for k_ν , and using $\nu_0 = \frac{2\pi^2 me^4}{n^2 h^3}$, we get

$$\frac{g_s}{g_0} f_c = n\bar{g} \times \frac{8}{3\pi\sqrt{3}}.$$

We do not put $g_s/g_0 = n^2$ because we are going to separate out the hydrogen lines into their several series.

The oscillator strengths for the hydrogen lines are given in Table I. It will be seen that they are given separately for each series S , P , D and F . For example, H_α appears as three separate transitions, $2S-3P$, $2P-3S$ and $2P-3D$. These values rest on values of the line-transition coefficients calculated by Sugiura § and Slack.||

To compare these separated values with Kramers' theory applied to the summed transitions we must weight the f 's and add for each transition. The result we have to test is

$$\sum \frac{g_s}{g_0} f_c = \bar{g} \times \frac{8n}{3\pi\sqrt{3}}.$$

Here $g_0 = 1$ and $g_s = 1, 3, 5, 7$ for S, P, D and F terms.

Table II shows the result of this test. The values of $\bar{g}$ are taken from Strömgren.¶ It will be seen that the agreement between the two methods of calculating k_ν is good. This agreement is simply a cross-check on quantum-mechanical calculations.

* *Zs. f. Astrophys.*, **4**, 118, 1932.

† *Veröff. Univ. Sternwarte, Göttingen*, Nr. 34.

‡ *Zs. f. Astrophys.*, **8**, 225, 1934.

§ Sugiura gives the f 's directly in *Sci. Papers Inst. Tokyo*, **11**, 1, 1929.

|| *Phys. Rev.*, **31**, 527, 1928.

¶ *Loc. cit.*

TABLE I
Oscillator Strengths of the Hydrogen Transitions

	Lyman Series	Balmer Series			Paschen Series				
Initial State Final State	$1S$ nP	$2S$ nP	$2P$ nS	$2P$ nD	$3S$ nP	$3P$ nS	$3P$ nD	$3D$ nP	$3D$ nF
$n = 1$	...	...	-0.139	...	...	-0.026	...	...	...
2	0.4162	...	...	...	-0.041	-0.142	...	-0.417	...
3	0.0791	0.425	0.014	0.694	...	...	...	...	...
4	0.0290	0.102	0.031	0.122	0.484	0.032	0.619	0.011	1.016
5	0.0139	0.042	0.012	0.044	0.121	0.007	0.139	0.0022	0.156
6	0.0078	0.022	0.006	0.020	0.052	0.003	0.056	0.009	0.053
7	0.0048	0.013	0.003	0.012	0.027	0.002	0.028	0.009	0.025
8	0.0032	0.008	0.002	0.008	0.016	0.001	0.017	0.002	0.015
$n = 9$ to ∞ together.	0.0101	0.026	0.007	0.023	0.048	0.002	0.045	0.007	0.037
Asymptotic formula.	$1.6n^{-3}$	$3.7n^{-3}$	$0.1n^{-3}$	$3.3n^{-3}$	$6.2n^{-3}$	$0.3n^{-3}$	$6.1n^{-3}$	$0.07n^{-3}$	$4.4n^{-3}$
Continuous spectrum.	0.4359	0.362	0.008	0.188	0.293	0.010	0.207	0.002	0.098
	1.0000	1.000	-0.111	1.111	1.000	-0.111	1.111	-0.400	1.400

These figures are taken from the *Handbuch d. Physik*, 24/1, p. 443. They are substantially the same as those given by Sugiura (*Sci. Papers Inst. Tokyo*, 11, 1, 1929). The transition probabilities given by Slack (*Phys. Rev.*, 31, 527, 1928) are systematically higher than Sugiura's by about 1 per cent.

TABLE II
Comparison of Continuous Absorption Coefficients

Series	n	Term	f Value for Continuous Spectrum f_c	$\Sigma q f_c/n$		Kramers' Theory $\frac{8}{3n\sqrt{3}} \times \bar{g}$	$\bar{g}$ Used *
				From Table I	From Sugiura		
Lyman	1	S	0.436	0.436	0.437	0.417	0.85
Balmer	2	S	0.362	0.475	0.457	0.442	0.90
		P	0.196				
Paschen	3	S	0.293	0.481	0.496	0.452	0.92
		P	0.217				
		D	0.100				

* These values have been obtained by weighted means of values in a table given by Strömgren, *op. cit.* Gaunt (*Phil. Trans. Roy. Soc.*, 229, 202, 1930) only states one significant figure in his values of g. The second figure given here is perhaps meaningless.

§ 2. It is when we pass on from hydrogen to more complicated atoms that the oscillator strength method of calculating continuous absorption coefficients is seen to the best advantage. Theoretical physics has not yet been able fully to grapple with the problem, whereas there is already a little experimental evidence about the numerical values of some f's in line tran-

sitions, and there is in addition the hope that experimental work may determine a number of f 's for line transitions in the near future. At the present time, the direct theory—Kramers' theory—has to suppose that the alkali metals are hydrogen-like. The formula for the absorption coefficient becomes

$$k(n, \nu) = \frac{64\pi^4}{3\sqrt{3}} \frac{Z^4 m e^{10}}{h^6 c^4 n^5} \times \bar{g} \lambda^3.$$

Ze being the nuclear charge. The principal quantum number n is given by

$$n^2 = \frac{2\pi^2 m e^4 Z^2}{\nu_0 h^3},$$

ν_0 being the frequency at the head of the series. These formulæ are used by Biermann. As an example, he gives the $3P$ level of Na . Here $Z=1$ and $h\nu_0=3.02$ volts, giving $n=2.12$.

Comparing the Kramers formula with the f_c formula and putting $\bar{g}=1$ (as is done throughout Biermann's work), we get

$$\frac{2\pi e^2 \nu_0^2 f_c}{m c^4} = \frac{64\pi^4}{3\sqrt{3}} \frac{Z^4 m e^{10}}{h^6 c^4 n^5},$$

or

$$f_c = \frac{8}{3\pi\sqrt{3}} \times \frac{1}{n} = \frac{0.617}{n}.$$

So that for the Na $3P$ level with $n=2.12$ we get $f_c=0.291$.

Let us turn to the ground state of Na . Here $h\nu_0=5.11$ volts from which we get $n=1.63$ and $f_c=0.474$.

This high value of f_c is inconsistent with Korff's experiments* on the laboratory absorption of the Na D lines. Korff found that the laboratory widths of these lines agree with the value given by the classical theory within 1 or 2 per cent. The sum of the f values of the lines is accordingly within 1 or 2 per cent. of unity, leaving very little to be distributed among all other transitions from the ground state, including the continuous spectrum. Heard† has found similar results for potassium. These experimental results indicate that *the Kramers formula applied to optical spectra even of the simplest metals gives too large a result*, unless one likes to save the Kramers formula by adopting values of g very different from unity.

§ 3. We go on to demonstrate the application of the oscillator strengths to another problem. Should we expect a sharp change in the coefficient of absorption as we pass over the head of a series? On the long wave-length side of the series limit the lines are crowded together at the head of the series, and overlap so as to form, effectively, a continuum; and on the short-wave side there is, of course, an actual continuum. Sugiura treats this point theoretically and finds no discontinuity. He quotes the experimental work of Herzberg‡ which showed that there was no discontinuity

* Korff, *Ap. J.*, **76**, 124, 1932.

† Heard, *M.N.*, **94**, 458, 1934.

‡ Herzberg, *Ann. d. Phys.*, **84**, 565, 1927.

at the head of the Balmer series, and also that of Trumpy * showing no discontinuity at the heads of series in *Na* and *Li*.

In the case of the hydrogen spectrum, if R is Rydberg's constant the frequency of the n th line in the m th series is

$$\nu = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$

and the interval in frequency between the n th and $(n+1)$ th lines is

$$R \left(\frac{2n+1}{n^2(n+1)^2} \right),$$

so that the number of lines in the interval $\Delta\nu$ near the head of the series is

$$\frac{n^2(n+1)^2}{(2n+1)} \cdot \frac{\Delta\nu}{R}.$$

For large n the density of the lines tends to become $\frac{n^3}{2R}\Delta\nu$.

Now the oscillator strengths of the lines in each hydrogen series tend towards an asymptotic value $f = \frac{\alpha}{n^3}$, where α is a constant characteristic of each series, so that the number of oscillators in the region $\Delta\nu$ tends to a limiting value $\frac{\alpha}{2R}\Delta\nu$. But ν_m , the frequency of the head of the m th series, is given by $m^2\nu_m = R$ so that the number of oscillators in the interval $\Delta\nu$ immediately on the long-wave side of the m th series limit is

$$\frac{\alpha}{2m^2} \times \frac{\Delta\nu}{\nu_m}.$$

On the short-wave side, on the other hand, the number of oscillators in the interval $\Delta\nu$ is $\phi_{\nu_m}\Delta\nu$, where $\phi_{\nu_m} = \frac{2f_c}{\nu_m}$. The oscillator strength per unit frequency interval is accordingly unchanged at the series limit if

$$\frac{\alpha}{4m^2} = f_c.$$

Table III shows the results for the Lyman, Balmer and Paschen series, set out for individual transitions, and also for the series as a whole. In the latter case we have to weight the individual transitions with the statistical weight of the lower state, viz. 1, 3, 5 for *S*, *P* and *D*.

It will be seen in the table that the figures for the series check up pretty well, though some sets of transitions (*e.g.* $2S - nP$) do not check at all. The failure to satisfy this check may mean a defect in the quantum-mechanical calculations on which Table I is based. But as far as the absorption coefficient in a stellar atmosphere is concerned, we may be satisfied that there is no sharp discontinuity at a series limit.

* Trumpy, *Zs. f. Phys.*, 47, 804, 1928 ; 54, 372, 1929

TABLE III

Continuity of Absorption Coefficients at Series Limits

Series	m	a	f_0	$a/4m^2$
$1S - nP$	1	1.6	0.436	0.400
$2S - nP$	2	3.7	0.362	0.231
$2P - nS$	2	0.1	0.008	0.007
$2P - nD$	2	3.3	0.188	0.206
Balmer Series (weighted mean of three above):				
			0.238	0.218
$3S - nP$	3	6.2	0.293	0.172
$3P - nS$	3	0.3	0.010	0.008
$3P - nD$	3	6.1	0.207	0.170
$3D - nP$	3	0.07	0.002	0.002
$3D - nF$	3	4.4	0.098	0.122
Paschen Series (weighted mean of five above):				
			0.160	0.158

Biermann* used Kramers' formula in his investigation of the solar absorption coefficient and was led to suppose that Na and Mg contribute materially to this coefficient. Minnaert† followed up Biermann's theoretical work by looking for discontinuities in the continuous solar spectrum at the head of the alkali series, but found none. We have already mentioned Trumphy's laboratory work which shows that there is no sharp discontinuity at the heads of these series, as the continuous spectrum shades off into a crowded line spectrum.

Apart from the laboratory evidence we have seen theoretically that there is no sharp discontinuity in the hydrogen series, and we may be tempted to generalise and suppose that there is no sudden discontinuity at the head of a series in any spectrum.

To conclude this discussion of a search for actual discontinuities in the continuous solar spectrum, we shall suggest in the next section that Fe , Si and Mg are the most important contributors to the solar coefficient of absorption after hydrogen: and we must note that there exists a mechanism (fluorescence) which tends to neutralise differences in the solar output of energy with wave-length consequent upon irregular variations in the absorption coefficient.

§ 4. It is known from Milne's researches that the coefficient of absorption of the solar atmosphere is, at any rate approximately, constant over the visible region of the spectrum. This constancy of k must arise through the balancing of hydrogen absorption by metallic absorption. The hydrogen absorption in the visible spectrum is absorption at the head of the Paschen series, and follows a λ^3 law from 8204 Å. to 3646 Å., the head of the Balmer series. Although the absorption at the head of each metallic series follows a λ^3 law, there are many more metallic series operating at 4000 Å. than at

* *Loc. cit.*† *B.A.N.*, 7, 234, 1934.

6000 Å., and the metallic absorption is greater in the violet than in the red. We now set ourselves the following problem. What must be the mean oscillator strength of the transitions in the continuous spectrum from an individual level in a metallic atom, if we are to have equal values for the coefficient of absorption in the solar atmosphere at 6000 Å. and at 4000 Å.?

Let there be N atoms per c.c. of a particular atom whose ionisation potential is I . Consider a level in this atom whose excitation potential is χ_s , and let $h\nu_s = I - \chi_s$.

Let the oscillator strength of the transitions in the continuous spectrum from this level be f_s ; and let the statistical weight of the level be g_s , that of the ground state of the atom being g_0 .

Let $\lambda_s = c/\nu_s$.

Then the contribution made by this level to k_λ is

$$0 \quad \text{if } \lambda > \lambda_s, \\ N g_s / g_0 \cdot \exp\left(-\frac{\chi_s}{RT}\right) \cdot \frac{2\pi e^2}{mc^4} \nu_s^2 f_s (\lambda/\lambda_s)^3 \quad \text{if } \lambda \leq \lambda_s.$$

Hence the absorption coefficient due to this element is

$$N \lambda^3 \frac{2\pi e^2}{mc^7 g_0} \sum g_s \exp\left(-\frac{\chi_s}{RT}\right) f_{c,s} \nu_s^5,$$

summed over all states s for which $\lambda_s \geq \lambda$, *i.e.* over all states whose excitation χ is subject to $\chi > I - h\nu$.

When we wish to compare metallic and hydrogen absorption we can omit the factor $\frac{2\pi e^2}{mc^7}$. The most convenient way of working up the result for some frequency ν between the Paschen and Balmer limits at 8204 Å. and 3646 Å. is to evaluate

$$N/g_0 \sum g_s f_s \exp\left(-\frac{\chi_s}{RT}\right) \frac{\nu_s^5}{\nu^3 \nu_H^2}, \quad (4.0)$$

where ν_H refers to 8204 Å. The coefficient of absorption due to hydrogen is then

$$N_H \cdot \left(\frac{g}{g_0}\right) f_H \exp\left(-\frac{\chi_H}{RT}\right) \frac{\nu_H^3}{\nu^3}. \quad (4.1)$$

For the Paschen series $\chi_H = 12.04$ volts and $\frac{\sum g f_H}{g_0} = 1.444$, from Table I.

Putting $f_{c,s} = f_c$, the mean metallic oscillator strength, we have for the metals, writing $E_s = I - \chi_s$,

At 6000 Å.:

$$k = N f_c / g_0 \sum g_s \exp\left(-\frac{\chi_s}{RT}\right) \times 0.5080 E_s^5 \text{ for all } E_s \leq 2.05 \text{ volts.} \quad (4.2)$$

At 4000 Å.:

$$k = N f_c / g \sum g_s \exp\left(-\frac{\chi_s}{RT}\right) \times 0.1506 E_s^5 \text{ for all } E_s \leq 3.08 \text{ volts.} \quad (4.3)$$

We use Russell's composition of the solar atmosphere. Values of N/g_0 are shown in Table IV. In calculating the ionisations, an electron pressure of 58 dynes/cm.² has been assumed.

TABLE IV
Russell's Composition of the Solar Atmosphere

Element	<i>H</i>	<i>He</i>	<i>O</i>	<i>K</i>	<i>Na</i>	<i>Al</i>	<i>Ca</i>	<i>Cr</i>	<i>Mg</i>	<i>Fe</i>	<i>Si</i>
<i>N/g</i> ₀	61,500	2000	2000	50	126	20	40	10	470	126	158
Neutral frac- tion 1 - α .	1	1	1	...	...	0.0078	0.0101	0.038	0.192	0.27	0.40
Ionised frac- tion α .	3.5 10 ⁻⁶	3.8 10 ⁻¹⁶	3.5 10 ⁻⁶	1	1	0.99	0.99	0.96	0.81	0.73	0.60
<i>I</i> , volts	13.54	24.47	13.56	4.32	5.12	5.96	6.09	6.74	7.61	7.83	8.12

It is clear at once that *Fe*, *Si* and *Mg* will be the most important contributors to the absorption. We might have expected something from the ionised metals, but their second ionisation potentials are all high, which means that the excitation potential χ_s must be large to get $E_s (= I - \chi_s)$ within the prescribed limit $E_s < 3.08$ volts. Hence all the terms due to ionised metals contain very small factors $\exp - (\chi_s/RT)$.

Actual calculations from (4.3) for *Si* are shown in Table V. The list of states is taken from Miss Moore's *Term Designations for Excitation Potentials*, and χ_s is taken from the same source. Then we have $E_s = I - \chi_s$. We have set 1.04 volts as a lower limit for E_s , as terms with a higher χ_s will contribute very little to the absorption.

TABLE V
Calculation of Si Absorption at Various Wave-lengths

State	<i>g</i> _s	χ_s	<i>E</i> _s	A	B	ΣAB	<i>k</i> × 10 ²
5 <i>p</i> ³ <i>P</i> ₁	3	7.08	1.04	0.054	0.018		
5 <i>p</i> ³ <i>P</i> ₀	1	7.07	1.05	0.18	0.19		
5 <i>p</i> ³ <i>D</i> ₃	7	7.06	1.06	0.132	0.20		
5 <i>p</i> ³ <i>D</i> ₂	5	7.04	1.08	0.097	0.22		
5 <i>p</i> ³ <i>D</i> ₁	3	7.04	1.08	0.058	0.22		
4 <i>p</i> ¹ <i>S</i> ₀	1	6.37	1.75	0.071	0.247		
3 <i>d</i> ³ <i>P</i> ₀	1	6.25	1.87	0.090	0.344		
3 <i>d</i> ³ <i>P</i> ₁	3	6.24	1.88	0.274	0.353		
3 <i>d</i> ³ <i>P</i> ₂	5	6.23	1.89	0.465	0.363		
4 <i>p</i> ¹ <i>P</i> ₁	3	6.21	1.91	0.290	0.383		
4 <i>p</i> ¹ <i>D</i> ₂	5	6.20	1.92	0.493	0.393		
4 <i>p</i> ³ <i>S</i> ₁	3	6.10	2.02	0.359	0.506		
4 <i>p</i> ³ <i>D</i> ₃	7	6.08	2.04	0.870	0.532		
4 <i>p</i> ³ <i>P</i> ₂	5	6.07	2.05	0.633	0.545	1.616	
6000 Å.				1.616 × (6000/4000) ³ = 5.454			2.22 × <i>f</i> ₀
4 <i>p</i> ³ <i>P</i> ₁	3	6.06	2.06	0.388	0.559		
4 <i>p</i> ³ <i>D</i> ₁	3	6.05	2.07	0.395	0.572		
4 <i>p</i> ³ <i>D</i> ₂	5	6.05	2.07	0.658	0.572		
4 <i>p</i> ³ <i>P</i> ₀	1	6.04	2.08	0.134	0.586	2.513	

State	g_s	χ_s	E_s	A	B	ΣAB	$k \times 10^3$
5750 A.			$2.513 \times (5750/4000)^3 = 7.464$				$3.03 \times f_c$
5500 A.			$2.513 \times (5500/4000)^3 = 6.433$				$2.61 \times f_c$
$3d^1D_2$	5	5.85	2.27	.969	.908	3.393	
5250 A.			$3.393 \times (5250/4000)^3 = 7.672$				$3.12 \times f_c$
5000 A.			$3.393 \times (5000/4000)^3 = 6.626$				$2.69 \times f_c$
$3d^3D_3^0$	7	5.60	2.52	2.199	1.530		
X^0	3?	5.59	2.53	0.960	1.562		
$3d^3D_2^0$	5	5.59	2.53	1.600	1.562		
$3d^3D_1^0$	3	5.59	2.53	0.960	1.562	12.256	
4750 A.			$12.256 \times (4750/4000)^3 = 20.529$				$8.34 \times f_c$
4500 A.			$12.256 \times (4500/4000)^3 = 17.452$				$7.09 \times f_c$
4250 A.			$12.256 \times (4250/4000)^3 = 13.727$				$5.58 \times f_c$
$4s^1P_1^0$	3	5.06	3.06	2.671	4.041	23.049	
4000 A.					23.049		$9.36 \times f_c$

The column A contains values of $g_0 \exp - \left(\frac{\chi_s - 5.0}{0.518} \right)$. The number 5 is subtracted from the list of χ_s for convenience in computing, as tables are given for small arguments of the exponential. The number 0.518 is the value of RT in volts for $T = 6000^\circ$. It is convenient to use $\frac{1}{0.518} = 1.9305$. Column B gives $0.01506 \times E_s^5$, and the next column ΣAB . This sum is taken at various points corresponding to 6000, 5750, 5500 A. down to 4000 A. Apart from the unknown f_c , the absorption is found from ΣAB by restoring the missing factor $\exp(-5.0/0.518)$ and multiplying by $N/g_0 \times (1-x) \times (\lambda/4000)^3$. The corresponding work for Fe is considerably heavier.

Collecting the results for 6000 A. and 4000 A. we have as under,

Element	N/g_0	$N/g_0(1-x)$	k at 6000 A.	No. of Levels Examined	k at 4000 A.	No. of Levels Examined
<i>Fe</i>	126	34	$0.1877f_c$	120	$0.9234f_c$	184
<i>Si</i>	158	63	$0.0222f_c$	14	$0.0936f_c$	24
<i>Mg</i>	470	90	$0.0182f_c$	8	$0.0302f_c$	10
Total			$0.2281f_c$	142	$1.0472f_c$	218

From (4.1) the hydrogen absorption in these units is 2.757×10^{-6} at 6000 A. and 0.817×10^{-6} at 4000 A. Equating the combined hydrogen and metallic absorption at these two wave-lengths, according to our hypothesis, we have

$$1.0472f_c + 0.817 \times 10^{-6} = 0.2281f_c + 2.757 \times 10^{-6},$$

giving

$$f_c = 2.37 \times 10^{-6}$$

for the mean oscillator strength for the continuous spectrum at the head of a metallic series, which will give the same coefficient of absorption in the solar atmosphere at 6000 Å. and 4000 Å. This very low value of f_c is an additional reason for supposing that Kramers' formula gives too high a result for optical continuous absorption. If Kramers' formula gives correct results, the solar absorption coefficient must be *very much* greater in the violet than in the red.

§ 5. We can now use this value of f_c in the absorption coefficients for the metals at various wave-lengths, the general formula being

$$k = f_c N(1-x)/g_0 \times \sum g_s \exp\left(-\frac{\chi_s}{RT}\right) \times 0.01506 E_s^5 \times (\lambda/4000)^3, \quad (5.1)$$

the sum being taken for all states such that $E_s < 3.08 \times \left(\frac{4000}{\lambda}\right)$, λ being in every case expressed in angstroms. To get the mean atomic absorption coefficient we should further multiply k by $2\pi e^2/mc^7$ and divide by 62,500, the number of atoms for which we have calculated the absorption.

Table VI gives results from (5.1) multiplied by 10^6 for convenience. (The factor $2\pi e^2/mc^7$ has not been applied.)

TABLE VI

Variation in Wave-lengths of the Solar Absorption Coefficient

Element	6000 Å.	5750 Å.	5500 Å.	5250 Å.	5000 Å.	4750 Å.	4500 Å.	4250 Å.	4000 Å.
<i>H</i>	2.757	2.426	2.092	1.847	1.596	1.368	1.163	0.915	0.817
<i>Fe</i>	0.445	.460	.672	.855	1.137	1.626	1.382	1.977	2.188
<i>Si</i>	0.053	.072	.062	.074	.064	.198	.168	.132	.221
<i>Mg</i>	0.043	.038	.047	.042	.036	.120	.102	.080	.072
Total	3.298	2.996	2.873	2.818	2.833	3.312	2.815	3.104	3.298

A noticeable feature is the high value of the absorption coefficient at 4750 Å. This is due to a particular crowding of series heads between 5000 Å. and 4750 Å. in all three spectra considered, *Fe*, *Si* and *Mg*. The variations in the absorption coefficient set out in Table VI must be real unless (1) our assumption that $k_{6000} = k_{4000}$ is altogether false; or (2) the oscillator strengths of the continuous spectra above different metallic series vary very widely, so that a mean value is untrustworthy, and some metal other than *Fe* is the principal contributor to the absorption coefficient.

We might look for irregularities in the solar absorption coefficient by examining the solar energy curve. A high value of the absorption coefficient at some wave-length gives a low output of energy at that wave-length. (To revive an old phrase, we do not see down into such deep, and therefore hot and bright, layers in this wave-length as we do in others.) But no irregularity in the observed energy curve is in fact seen at 4750 Å. of a size corresponding to the variation in the absorption coefficient shown in Table VI.

If neither objection (1) nor (2) is valid, we must suppose that there is some compensating mechanism tending to counteract an irregularity in the absorption coefficient in its effect on the solar energy curve.

Such a mechanism is supplied by fluorescence. Let us consider the cyclical changes which may go on between three states in an atom, state 1, the ground state; state 2, a state of high excitation; and state 5,* the ionised state. The cycle 1251 takes place by absorption of one line quantum and one quantum in a continuous spectrum, and the emission of one quantum in another continuous spectrum of shorter wave-length than the first. (The energy of the quantum 51 is the sum of the energies 12+25.) The cycle 1521 takes place by the absorption of the short-wave quantum and the emission of the other two. Now the radiation at the Sun's surface is dilute temperature radiation, and, according to Rosseland's theorem,† the cycle 1521 will take place more often than the cycle 1251. The process is discussed in a paper by the writer‡ in which figures are given for hydrogen transitions in the Sun. In the notation of that paper the quantity α is the ratio of the number of transitions 52 to the transitions 25, and it can be shown in this case, in which only three states 1S, 2P and the ionised state are considered, to be given by

$$\alpha = \frac{A_{52}}{A_{55}} \left(1 + \frac{1}{W} \frac{A_{21}A_{15}}{A_{25}A_{12}} \right).$$

If $W = \frac{1}{3}$, $\alpha = 2.85$ at the solar temperature and the atmosphere is strongly fluorescent in the transitions 52, the continuous spectrum beyond the head of the Balmer series. §

The output of fluorescent radiation at any wave-length in a continuous spectrum is proportional to the oscillator strength at that point (or nearly so), so that the fluorescent return to any wave-length made by each series is very nearly proportional to its contribution to the absorption coefficient at that point. Hence a hole in the output of solar energy such as that at 4750 Å. would *tend* to fill up, on account of the fluorescence. We know, of course, from the fact that Balmer continuous absorption is seen in early type stars that the balance between fluorescence and absorption is not, in general, complete.

All writers on the subject are agreed that in the spectra of stars of types Bo to Go there exists some kind of discontinuity in the energy curve at the head of the Balmer series. According to C. S. Yü, || who made a number of observations of the energy curves in stars of type B to G using the star ζ Ophiuchi (type Bo) as a standard (presumed to be radiating like a black body at 22,000°), nearly all stars show *absorption* in the continuous spectrum

* The notation is kept the same as that of an earlier paper by the writer (*M.N.*, 94, 631, 1934).

† *Ap. J.*, 63, 218, 1926.

‡ *M.N.*, 94, 631, 1934.

§ Notice that the fluorescence is stronger for the continuous spectrum than for the lines in the Balmer series, *e.g.* under the same conditions $\alpha = 1.17$ for H_β .

|| *L.O.B.*, 12, 104, 155, 1926.

on the short-wave side of the Balmer limit: but there is a good deal of evidence to show that in certain early type stars there is an *increase* in the energy output on the short-wave side of this limit.* Gerasimovič † draws attention to "a very considerable strengthening of the continuous spectrum beyond the Balmer limit for B stars." He describes this feature as the "ultra-violet appendage," and states that it is especially strong in spectra of classes O, Bo and B2, that it is present in *all* spectra of these classes, and is very strong for the abnormally yellow stars such as P Cygni and ζ Persei. Kienle ‡ compared the two stars of class B1, ζ Persei and ϵ Persei, and found that while ζ is definitely yellower than ϵ on the long-wave side of the Balmer limit, there is a discontinuity at the limit, the radiation from ζ on the short-wave side of the limit being much enhanced. Again, C. S. Yü § calls attention to continuous ultra-violet emission in the spectra of certain Be stars. ||

This experimental evidence demonstrates the existence of continuous fluorescent emission on the short-wave side of the Balmer limit, and shows that it can compete with the ordinary absorption and sometimes master it, so that our theoretical surmise that fluorescent emission would tend to neutralise an increase in the absorption coefficient receives observational support. It is very tempting to go a step further and speculate whether the abnormal yellow colour of ζ Persei is not due to fluorescent emission coming to a maximum at the head of the Paschen series. If this is so, we should expect to see another discontinuity in the energy curve near the limit of the Paschen series.

Summary.—We suppose that the continuous absorption in the solar atmosphere takes place as a result of absorption in the continuous spectra at the heads of the various line series. With the help of the oscillator strengths, it is possible to calculate the continuous absorption from the line intensities. The results for hydrogen (the only completely calculable case) agree with Kramers' theory. The continuous absorption for Na calculated from Kramers' theory implies an oscillator strength which appears to be in complete disagreement with laboratory measurements of the D lines. Next, the mean oscillator strength for metallic transitions effective in the solar atmosphere is calculated by assuming that metallic absorption balances hydrogen absorption in such a way as to make $k_{6000 \text{ \AA}} = k_{4000 \text{ \AA}}$. The result is f_c , the sum of the oscillator strengths in the continuous spectrum above each metallic series, equals 2.37×10^{-6} . The principal contributors to the absorption (after H) are Fe, Si and Mg. Using this figure the variation of

* The writer is much indebted to Mr. W. M. H. Greaves for calling his attention to this point.

† *Harvard Circ.*, 339.

‡ *M.N.*, 88, 700, 1928.

§ *P.A.S.P.*, 39, 112, 1927.

|| The views of Miss Payne (*Stars of High Luminosity*, p. 118) that "there is no appendage in the ultra-violet, the spectra suffer a 'violet depression'" are altogether at variance with the views on the nature of absorption expressed in this paper, if the "depression" is supposed to be due to a high value of the absorption coefficient in the violet which disappears in the ultra-violet (at the Balmer limit). We must take Miss Payne's "depression" to be an *absence of fluorescence* in the violet.

the coefficient of absorption of the solar atmosphere with wave-length is calculated. Irregularities appear in this coefficient. It is shown theoretically that no *abrupt* change in absorption coefficient takes place at the head of a series, and finally as regards cumulative effects it is shown that fluorescence acts in such a manner as to neutralise irregularities in the absorption coefficient. The bearing of the "ultra-violet appendages" in certain B stars on these problems is briefly discussed.

SOME THEOREMS IN THE STATISTICAL TREATMENT OF STELLAR MOTIONS.

W. M. Smart, *M.A., D.Sc.*

1. In a recent paper F. K. Edmondson has derived the formula *

$$\bar{p} = 3 \cdot 016 \frac{\sum |\mu|}{\sum |R|}$$

for the mean parallax $\bar{p}$ of stars evenly distributed over the celestial sphere, where μ is the total annual proper motion of a star (in seconds of arc) *due to its peculiar tangential linear velocity*, and R is its *peculiar radial velocity* in km. per second. It is assumed that the peculiar linear space velocities of the stars have a random distribution. We shall write the formula in terms of the mean motions as

$$\bar{p} = 3 \cdot 016 \frac{\bar{\mu}}{\bar{R}}, \quad (1)$$

in which μ and R are considered without regard to sign.

The formula is only strictly true if the stars concerned are at the same distance from the Sun, but its use may be justified in statistical investigations for groups of stars with a limited magnitude range and of similar spectral characteristics.

Edmondson then infers that the effects of the solar motion in forming the sums $\sum |\mu_0|$ and $\sum |R_0|$, where μ_0 and R_0 are the *observed* proper motion and *observed* radial velocity of a star, will cancel out when the summation is taken over the sphere; he then writes

$$\bar{p} = 3 \cdot 016 \frac{\sum |\mu_0|}{\sum |R_0|},$$

or, in terms of the mean motions, taken regardless of sign,

$$\bar{p} = 3 \cdot 016 \frac{\bar{\mu}_0}{\bar{R}_0}. \quad (2)$$

* *A.J.*, 42, 22, 1932.

This formula is the starting-point of a series of papers * by Edmondson and J. M. Mohr. In particular, Edmondson applies (2) to the various magnitude groups of stars listed in Schlesinger's *Catalogue of Bright Stars*, and from his results finds justification in the belief that the method is reliable.

From the computer's point of view (2) has the merit of extreme simplicity, but unfortunately the arguments adduced in deriving (2) from (1) are erroneous.

In the present paper I investigate several problems associated with the mean proper motions and radial velocities of the stars. The consideration of preferential motion may be undertaken on the basis of the two-streams theory or on the ellipsoidal theory; it appeared at first sight that the former theory would probably be simpler analytically, and it has accordingly been adopted in the sequel.

We consider that the peculiar or haphazard velocities of the stars forming a drift are distributed in such a way that the number with linear velocity-components between (u, v, w) and $(u + du, v + dv, w + dw)$ is proportional to

$$F(W)dudvdw,$$

where

$$W^2 = u^2 + v^2 + w^2 \quad (3)$$

and $F(W)$ is an even function of W . This distribution of velocities will be called a *spherical velocity-distribution*. In particular a Maxwellian distribution of velocities, as in Eddington's theory,† will be considered, and in this case

$$F(W) = e^{-h^2 W^2}, \quad (4)$$

where h is a constant associated with the mean random velocity.

Sections (2) and (3) contain brief derivations, on well-known lines, of formulæ required in the subsequent analysis.

2. Consider a small area of the celestial sphere at S , in which there are N stars per unit area with the given spherical velocity-distribution. Let the w -axis at S be radial. Then w is the peculiar radial velocity R , and the peculiar tangential linear velocity T is given by

$$T = (u^2 + v^2)^{\frac{1}{2}}. \quad (5)$$

If dN is the number of stars per unit area at S with velocity components between (u, v, w) and $(u + du, v + dv, w + dw)$, we can write

$$dN = BF(W)dudvdw, \quad (6)$$

where B is a constant if the distribution over the sphere is uniform, or a function of the galactic co-ordinates in the more general case of non-uniform distribution.

From (6),

$$N = B \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(W)dudvdw$$

* *A.J.*, 42, 95, 1932; 42, 157, 1933; 43, 92, 1934; 44, 10 and 15, 1934.

† *M.N.*, 67, 34, 1906.

or, since F is an even function of W ,

$$N = 8B \int_0^\infty \int_0^\infty \int_0^\infty F(W) du dv dw. \quad (7)$$

Transforming to polar co-ordinates by means of

$$u = W \cos \phi \sin \theta, \quad v = W \sin \phi \sin \theta, \quad w = W \cos \theta, \quad (8)$$

we have

$$N = 8B \int_0^\infty W^2 F(W) dW \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\phi$$

or

$$N = 4\pi B Q_1, \quad (9)$$

where

$$Q_1 = \int_0^\infty W^2 F(W) dW. \quad (10)$$

In the same way, if $\bar{R}$ denotes the mean arithmetical value of R ($\equiv w$) for the stars concerned,

$$N\bar{R} = 2B \int_0^\infty w dw \int_{-\infty}^\infty \int_{-\infty}^\infty F(W) du dv$$

which, by the transformation (8), becomes

$$N\bar{R} = 8B \int_0^\infty W^3 F(W) dW \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{\pi/2} d\phi$$

or

$$N\bar{R} = 2\pi B Q_2, \quad (11)$$

where

$$Q_2 = \int_0^\infty W^3 F(W) dW. \quad (12)$$

From (9) and (11) we obtain

$$\bar{R} = \frac{Q_2}{2Q_1}. \quad (13)$$

It is to be noted that Q_1 and Q_2 depend only on the form of velocity-distribution function F . Consequently, the value of $\bar{R}$ is constant for all areas of the celestial sphere, and is independent of the functional form of B . Thus the mean arithmetical random radial velocity for all stars distributed over the celestial sphere in any way is given by formula (13).

3. Consider now the random linear tangential velocity T , given by (5). If $\bar{T}$ denotes the mean value of $(u^2 + v^2)^{1/2}$, taken without regard to sign, for the stars in the area of the celestial sphere at S , we have from (6)

$$N\bar{T} = 4B \int_{-\infty}^\infty dw \int_0^\infty \int_0^\infty (u^2 + v^2)^{1/2} F(W) du dv.$$

Using the transformation (8) we obtain

$$N\bar{T} = 8B \int_0^\infty W^3 F(W) dW \int_0^{\pi/2} \sin^2 \theta d\theta \int_0^{\pi/2} d\phi,$$

from which

$$NT = B\pi^2 Q_2,$$

and by (9),

$$\bar{T} = \frac{\pi Q_2}{4Q_1}. \quad (14)$$

As for $\bar{R}$, this value of $\bar{T}$ depends only on the form of the function F , and is applicable to stars distributed in any way over the celestial sphere.

From (13) and (14) we obtain

$$T = \frac{\pi}{2} \bar{R}, \quad (15)$$

and this result is independent of the frequency function F .

4. If p is the parallax of a star and μ is the peculiar annual proper motion corresponding to the peculiar linear tangential velocity T , we have

$$T = 4.74 \frac{\mu}{p}, \quad (16)$$

where p and μ are measured in seconds of arc and linear velocities in km. per second.

If $\bar{\mu}$ denotes the mean peculiar proper motion, it follows that

$$\bar{T} = 4.74 \frac{\bar{\mu}}{\bar{p}} \quad (17)$$

for stars at the same distance. In a statistical investigation of stars with magnitudes between m and $m+1$, say, and of a given spectral class, we can write (17) in the form

$$\bar{T} = 4.74 \frac{\bar{\mu}}{\bar{p}}, \quad (18)$$

where $\bar{p}$ is the mean parallax of the stars concerned.

From (15) we obtain

$$\bar{p} = 3.016 \frac{\bar{\mu}}{\bar{R}}, \quad (19)$$

which is equivalent to Edmondson's formula (1). As in the case of (15), formula (19) is independent of the frequency function F .

5. If the mean random total linear velocity of the stars is denoted by $\bar{W}$, we have, as in the previous sections,

$$N\bar{W} = 8B \int_0^\infty WF(W)dw \int_0^\infty \int_0^\infty dudv,$$

which gives, using (9) and (12),

$$\bar{W} = \frac{Q_2}{Q_1}. \quad (20)$$

Hence, irrespective of the form of the function F , we have

$$\bar{R} = \frac{1}{2} \bar{W} \quad (21)$$

and

$$\bar{T} = \frac{\pi}{4} \bar{W}. \quad (22)$$

In the case of a Maxwellian distribution of linear velocities F is given by (4), and it is easily found that

$$Q_1 = \frac{\sqrt{\pi}}{4h^3} \quad \text{and} \quad Q_2 = \frac{1}{2h^4}. \quad (23)$$

Hence the well-known formulæ are obtained :

$$\bar{R} = \frac{1}{h\sqrt{\pi}}, \quad \bar{T} = \frac{\sqrt{\pi}}{2h} \quad \text{and} \quad \bar{W} = \frac{2}{h\sqrt{\pi}}, \quad (24)$$

and any one of these may be taken to define the constant h .

6. We now consider an assembly of stars, with a spherical velocity-distribution, with reference to which the solar velocity is $-U$. Let R_0

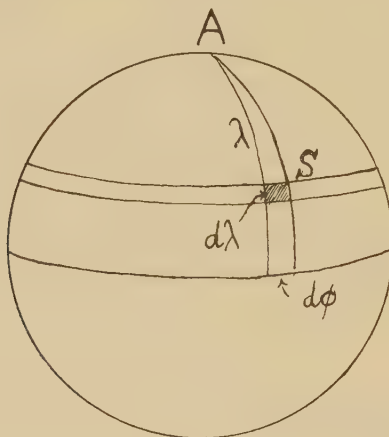


FIG. 1.

denote the *observed* radial velocity with reference to the Sun ; R_0 will of course differ from the peculiar radial velocity R of the star by an amount depending on the solar motion and the star's position on the celestial sphere.

We proceed to obtain an expression for $\bar{R}_0$, the individual values of R_0 being taken without regard to sign.

In fig. 1 let A be the antapex of the solar motion. Consider a small area dA at S defined by the small circles at angular distances λ and $\lambda + d\lambda$ from A and between the "meridians" ϕ and $\phi + d\phi$. Then

$$dA = \sin \lambda \, d\lambda \, d\phi. \quad (25)$$

We then have for any star at S ,

$$R_0 = R + U \cos \lambda;$$

or, putting $V \equiv U \cos \lambda$,

$$R_0 = R + V. \quad (26)$$

If N is the number of stars per unit area at S , the number dN with peculiar radial velocities (algebraic) between R and $R + dR$ is given by

$$dN = BdR \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(W) du dv,$$

which can be expressed as

$$dN = CG(R)dR, \quad (27)$$

where $G(R)$ is an even function of R , and

$$N = 2C \int_0^{\infty} G(R)dR. \quad (28)$$

For a uniform distribution of stars over the celestial sphere C will be a constant; otherwise C will depend on the position of S and, in the general case, can be regarded as a function of the galactic co-ordinates of S .

7. We first consider the hemisphere for which $0 < \lambda < 90^\circ$. In this case V in formula (26) is a positive quantity.

Let $2N_1$ denote the number of stars per unit area at S with radial velocities such that $0 \leq |R| \leq V$ and $2N_2$ the number for which $|R| > V$. Then

$$N = 2(N_1 + N_2), \quad (29)$$

and from (27),

$$N_1 = C \int_0^V G(R)dR. \quad (30)$$

We now form $\Sigma |R_0|$ for unit area at S .

The N stars are divided into three groups:

- (i) $N_1 + N_2$ stars for each of which R is positive and $0 < R < \infty$.
 R_0 is accordingly positive and

$$|R_0| = |R| + V,$$

so that

$$\sum_{N_1 + N_2} |R_0| = \sum_{R=0}^{\infty} |R| + (N_1 + N_2)V.$$

- (ii) N_1 stars for each of which R is negative and $0 < |R| \leq V$.
 R_0 is accordingly positive and

$$|R_0| = -|R| + V,$$

so that

$$\sum_{N_1} |R_0| = - \sum_0^V |R| + N_1 V.$$

- (iii) N_2 stars for each of which R is negative and $|R| > V$.
 R_0 is therefore negative and

$$-|R_0| = -|R| + V,$$

so that

$$\sum_{N_2} |R_0| = \sum_V^{\infty} |R| - N_2 V.$$

Hence, adding the results in (i), (ii) and (iii), we have for the N stars per unit area at S ,

$$\sum_N |R_0| = 2 \sum_V |R| + 2N_1 V; \quad (31)$$

or, if R_0' denotes the mean value of $|R_0|$,

$$NR_0' = 2C \int_V^\infty RG(R) dR + 2CV \int_0^V G(R) dR,$$

using (27) and (30).

This last equation can be written:

$$NR_0' = 2C \int_0^\infty RG(R) dR + 2C \int_0^V (V - R)G(R) dR. \quad (32)$$

But the mean peculiar radial velocity $\bar{R}$, the individual random radial velocities being taken without regard to sign, is given by

$$N\bar{R} = 2C \int_0^\infty RG(R) dR, \quad (33)$$

and if we write

$$Q = \int_0^\infty G(R) dR, \quad (34)$$

formula (32) becomes, using (28),

$$R_0' = \bar{R} + \frac{1}{Q} \int_0^V (V - R)G(R) dR. \quad (35)$$

The mean peculiar velocity $\bar{R}$ is that investigated in section 2.

8. Consider now a small area at S' , the point on the sphere antipodal to S . If we assume symmetry of distribution of the stars with reference to the galactic equator, the number of stars per unit area at S' will be the same as at the antipodal point S . The observed radial velocity will now be given by

$$R_0 = R - V, \quad (36)$$

in which V is positive. We have the following groups into which the N stars per unit area at S' are divided:—

- (i) N_2 stars for each of which $R > V$.

Then R_0 is positive and

$$\sum_{N_2} |R_0| = \sum_V |R| - N_2 V.$$

- (ii) N_1 stars for each of which R is positive and $0 < R < V$.

Then R_0 is negative and

$$-|R_0| = |R| - V,$$

and consequently,

$$\sum_{N_1} |R_0| = - \sum_0^V |R| + N_1 V.$$

- (iii) $N_1 + N_2$ stars for each of which R is negative and $0 < |R| < \infty$.
Then R_0 is negative and

$$-|R_0| = -|R| - V,$$

and therefore

$$\sum_{N_1+N_2} |R_0| = \sum_0^{\infty} |R| + (N_1 + N_2)V.$$

Summing the results in (i), (ii) and (iii), we obtain formula (31). The results for unit areas at the antipodal points are thus the same, and in the sequel we need consider only one of the hemispheres of which the antapex A is the pole.

9. The number of stars in the area dA at S is NdA , where dA is given by (25). Hence integrating over the hemisphere, we have from (35), and using (28) and (34),

$$\int R_0' NdA = \int \bar{R} NdA + 2 \int C dA \int_0^V (V - R) G(R) dR.$$

Let N_0 be the total number of stars over the hemisphere. Let $\bar{R}_0$ be the mean of the arithmetical values of the observed radial velocities R_0 for the hemisphere. Then

$$N_0 \bar{R}_0 = N_0 \bar{R} + 2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} C \sin \lambda d\lambda d\phi \int_0^V (V - R) G(R) dR, \quad (37)$$

for, as we have already shown, $\bar{R}$ is the same for all positions on the sphere. In (37) C will be a function of λ and ϕ in the general case, but the integral is hardly likely to lead to simple treatment. Accordingly, we regard C as a constant. The procedure in any statistical investigation must then be modified by considering only the means of the observed quantities over each small area of the sphere, irrespective of the number of stars utilised in such areas. The constancy of C implies—from (28)—that the number N of stars per unit area is also constant. Hence $N_0 = 2\pi N$. Also $N = 2CQ$. Formula (37) becomes

$$\bar{R}_0 = \bar{R} + \frac{1}{Q} \int_0^{\frac{\pi}{2}} \sin \lambda d\lambda \int_0^V (V - R) G(R) dR. \quad (38)$$

Now V is a function of λ —we have $V = U \cos \lambda$ —hence (38) becomes

$$\bar{R}_0 = \bar{R} + \frac{1}{QU} \int_0^U dV \int_0^V (V - R) G(R) dR;$$

or, changing the order of integration in the double integral,

$$\bar{R}_0 = \bar{R} + \frac{1}{QU} \int_0^U G(R) dR \int_R^U (V - R) dV,$$

whence

$$\bar{R}_0 = \bar{R} + \frac{1}{2QU} \int_0^U (U - R)^2 G(R) dR. \quad (39)$$

10. We now derive the formula for $\bar{R}_0$ when the assembly of stars has a

Maxwellian velocity-distribution. In this case, $G(R) = e^{-h^2 R^2}$, and from the first equation of (24)

$$\bar{R} = \frac{1}{h\sqrt{\pi}}. \quad (40)$$

Also from (34),

$$Q = \frac{\sqrt{\pi}}{2h}. \quad (41)$$

Now (39) becomes

$$\bar{R}_0 = \bar{R} + \frac{1}{2QU} \int_0^U (U-R)^2 e^{-h^2 R^2} dR. \quad (42)$$

Write

$$hR = x \quad \text{and} \quad hU = \tau. \quad (43)$$

Then from (42),

$$\frac{\bar{R}_0}{\bar{R}} = 1 + \frac{1}{\tau} \int_0^\tau (\tau - x)^2 e^{-x^2} dx; \quad (44)$$

or, writing

$$K(\tau) = \int_0^\tau e^{-x^2} dx, \quad (45)$$

we obtain

$$\bar{R}_0 = \bar{R}\phi(\tau), \quad (46)$$

where

$$\phi(\tau) = \left(\tau + \frac{1}{2\tau} \right) K(\tau) + \frac{1}{2} e^{-\tau^2}. \quad (47)$$

The values of the function $\phi(\tau)$ are given in Table I.

TABLE I

Values of $\phi(\tau)$

τ	$\phi(\tau)$	τ	$\phi(\tau)$
0.4	1.052	1.2	1.423
0.5	1.081	1.3	1.487
0.6	1.116	1.4	1.553
0.7	1.156	1.5	1.622
0.8	1.201	1.6	1.693
0.9	1.250	1.7	1.766
1.0	1.304	1.8	1.841
1.1	1.362	1.9	1.917
1.2	1.423	2.0	1.994

11. The results of the previous section refer to a single drift of stars, and we now derive the appropriate formulæ on the basis of the two-streams theory. Assume that the numbers of the stars in the two drifts are distributed in the proportions α and $(1 - \alpha)$. Let the linear velocities of the drifts be U_1 and U_2 with respect to the Sun, and write $\tau_1 = hU_1$, $\tau_2 = hU_2$. From (46) we have

$$\alpha \bar{R}_0' + (1 - \alpha) \bar{R}_0'' = \bar{R} \{ \alpha \phi(\tau_1) + (1 - \alpha) \phi(\tau_2) \},$$

in which $\bar{R}_0'$ and $\bar{R}_0''$ refer to the mean arithmetical observed radial velocities in drift I and drift II respectively for the whole sphere. Denoting by $\bar{R}_0$ the mean arithmetical radial velocity for all the stars observed, we obtain

$$\bar{R}_0 = f_1 \bar{R}, \quad (48)$$

where

$$f_1 = \alpha \phi(\tau_1) + (1 - \alpha) \phi(\tau_2). \quad (49)$$

For the Boss stars, Eddington's results * are

$$\tau_1 = 1.516, \quad \tau_2 = 0.855 \quad \text{and} \quad \alpha = 0.6.$$

From Table I, $\phi(\tau_1) = 1.632$ and $\phi(\tau_2) = 1.226$. Hence $f_1 = 1.47$, so that

$$\bar{R}_0 = 1.47 \bar{R}.$$

My own results,† derived from the Cambridge photographic proper motions, are

$$\tau_1 = 1.497, \quad \tau_2 = 0.975 \quad \text{and} \quad \alpha = 0.5.$$

For these stars, mainly of magnitude 9 to 12, the relation is found to be

$$\bar{R}_0 = 1.46 \bar{R}.$$

It follows that, due to the relation (48) alone, all mean parallaxes determined by Edmondson have to be multiplied by the factor f_1 , whose approximate value is 1.5.

12. Formula (48) has an important application in the two-streams theory, in which all linear velocities are expressed in terms of the theoretical unit $1/h$. If the drift velocities U_1 and U_2 are to be expressed in terms of familiar units, we must first find the value of $1/h$ in terms of these units. This may be done by equating the theoretical value of the solar motion, derived in the two-streams theory, to the value derived from an investigation of the solar motion based on radial velocity measures alone. From the Boss stars, for example, Eddington found ‡ the value 0.908 (in terms of the unit $1/h$) for the solar motion; if this is equated to the value of 19.5 km. per sec. obtained from the radial velocities, the theoretical unit is thus about 21.5 km. per sec. But, as Eddington then remarked, this procedure is inadvisable owing to the marked discordance between the positions of the solar apex determined from the radial motions and from the proper motions. So far as the bright stars are concerned, the gap (mainly in declination) between the two positions of the solar apex has been considerably decreased in recent years; the declination found § in 1928 by Campbell and Moore from radial motions is 29° as compared with Eddington's value of 36° . If, as is possible, the discrepancy arises from an insufficient number of radial velocities, the above method of deducing the value of the theoretical value $1/h$ must be set aside until the radial velocity material is adequate. This objection has less weight in connection with the use of formula (48); with 3000 radial velocity measures and the constants of the star streams known, it should be possible to determine $\bar{R}$ (in km. per sec.). But

* *M.N.*, 71, 35, 1910.

† *Ibid.*, 87, 135, 1926.

‡ *Ibid.*, 71, 35, 1910.

§ *Lick Publ.*, 16, 1928.

$\bar{R} = 1/h\sqrt{\pi}$ —from (24)—hence $1/h$ can be found. It is to be noticed that this method of determining $1/h$ does not depend on the solar motion as derived from radial velocities.

13. We now investigate the tangential linear velocities of an assembly of stars with a Maxwellian distribution of random velocities. As before, N is the number of stars per unit area at S (fig. 1). Let V now denote * the projection of the drift velocity, with respect to the Sun, on the tangential plane at S , and let the position angle θ be measured from this line. A star with tangential random components (u, v) will then have a total tangential linear velocity r in position angle θ , and the number, dN , of stars with tangential linear velocities relative to the Sun between position angles θ and $\theta + d\theta$ is given, as shown † by Eddington, by

$$dN = \frac{Nh^2}{\pi} d\theta \int_0^\infty e^{-h^2(r^2 - 2Vr \cos \theta + V^2)} r dr;$$

or, putting $x = h(r - V \cos \theta)$ and $\tau = hV \cos \theta$,

$$dN = Bf(\tau)d\theta, \quad (50)$$

where

$$B = \frac{N}{2\sqrt{\pi}} e^{-\frac{1}{2}h^2 V^2} \quad (51)$$

and

$$f(\tau) = \frac{2}{\sqrt{\pi}} \left\{ \frac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^\infty e^{-x^2} dx \right\}. \quad (52)$$

Further, Eddington has shown (*loc. cit.*, p. 56) that the mean value $\bar{r}$ for stars with tangential velocities, relative to the Sun, between position angles θ and $\theta + d\theta$ is given by

$$h\bar{r} = \frac{\frac{1}{\sqrt{\pi}} e^{\tau^2} \int_{-\tau}^\infty e^{-x^2} dx}{f(\tau)} + \tau. \quad (53)$$

By (50), the number of stars concerned is $Bf(\tau)d\theta$. Hence, if $\bar{r}_0$ denotes the mean value of r for the N stars between position angles 0 and 2π ,

$$\begin{aligned} N\bar{r}_0 &= B \int_0^{2\pi} f(\tau) \bar{r} d\theta \\ &= 2B \int_0^\pi f(\tau) \bar{r} d\theta, \end{aligned}$$

since the drift curve is symmetrical about $\theta = 0$. Using (52) and (53) we obtain

$$\begin{aligned} N\bar{r}_0 &= \frac{2B}{h\sqrt{\pi}} \int_0^\pi \left\{ \tau + (1 + 2\tau^2) e^{\tau^2} \int_{-\tau}^\infty e^{-x^2} dx \right\} d\theta \\ &= \frac{2B}{h\sqrt{\pi}} \int_0^\pi \left[\tau + (1 + 2\tau^2) e^{\tau^2} \left\{ \frac{\sqrt{\pi}}{2} + K(\tau) \right\} \right] d\theta \end{aligned}$$

* Not to be confused with the “ V ” of sections 6–9.

† *M.N.*, 67, 36, 1906.

by (45). Now $K(\tau)$ is an odd function of $\tau \equiv hV \cos \theta$, and we have, inserting the value of B given by (51),

$$\bar{r}_0 = \frac{1}{2h\sqrt{\pi}} e^{-h^2 V^2} \int_0^\pi (1 + 2\tau^2) e^{\tau^2} d\theta.$$

We shall write

$$2b = (hV)^2, \quad (54)$$

so that

$$\tau = \sqrt{2b} \cos \theta.$$

Hence

$$\bar{r}_0 = \frac{1}{2h\sqrt{\pi}} \int_0^\pi e^{-2b \sin^2 \theta} (1 + 4b \cos^2 \theta) d\theta; \quad (55)$$

or, in the alternative form,

$$\bar{r}_0 = \frac{1}{2h\sqrt{\pi}} e^{-b} \int_0^\pi e^{b \cos 2\theta} (1 + 2b + 2b \cos 2\theta) d\theta. \quad (56)$$

The integral on the right of (56) can be expressed in terms of Bessel functions of imaginary argument, since

$$\int_0^\pi e^{b \cos 2\theta} d\theta = \pi \mathfrak{J}_0(ib)$$

and

$$\int_0^\pi \cos 2\theta e^{b \cos 2\theta} d\theta = -i\pi \mathfrak{J}_1(ib) \equiv \pi \frac{d\mathfrak{J}_0(ib)}{db}.$$

Hence

$$\bar{r}_0 = \frac{\sqrt{\pi}}{2h} e^{-b} \{(1 + 2b)\mathfrak{J}_0(ib) - 2ib\mathfrak{J}_1(ib)\}.$$

Write

$$e^{-\frac{1}{2}n\pi i} \mathfrak{J}_n(ib) \equiv I_n(b).$$

The function $I_n(b)$ is real. Then

$$\bar{r}_0 = \frac{\sqrt{\pi}}{2h} e^{-b} \{(1 + 2b)I_0(b) + 2bI_1(b)\}. \quad (57)$$

By (24), the mean transverse random velocity $\bar{T}$ is $\sqrt{\pi}/2h$. Hence we can write

$$\bar{r}_0 = \bar{T}\psi(b), \quad (58)$$

where

$$\psi(b) = e^{-b} \{(1 + 2b)I_0(b) + 2bI_1(b)\}. \quad (59)$$

The values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are tabulated in G. N. Watson's *Theory of Bessel Functions*, pp. 698–713, 1922.

Let now $b_1 \equiv \frac{1}{2}(hV_1)^2$ refer to drift I and $b_2 \equiv \frac{1}{2}(hV_2)^2$ refer to drift II, for the area at S in fig. 1. If the drift constants are known for this area, $\psi(b_1)$ and $\psi(b_2)$ can be calculated. If the numbers of stars in the two drifts

are distributed in the proportions α and $(1 - \alpha)$, the mean transverse velocity $\bar{\rho}_0$ for all the stars in the region is accordingly given by

$$\bar{\rho}_0 = \bar{T} \{ \alpha \psi(b_1) + (1 - \alpha) \psi(b_2) \}. \quad (60)$$

In section (12) a method of expressing $1/h$ in terms of the usual units was indicated. If this is assumed done, $\bar{T}$ can then be found in km./sec.

If we assume for the moment that all the stars considered in the area at S have the same parallax p , we have

$$\bar{\rho}_0 = 4.74 \frac{\bar{\mu}_0}{p}, \quad (61)$$

where $\bar{\mu}_0$ is the mean total observed proper motion for all the stars. We thus obtain an expression for p in terms of $\bar{\mu}_0$, $\bar{T}$, α and the ψ functions. In any practical application to stars of a particular magnitude range in a small region of the sky, the formulæ (60) and (61) will yield a statistical mean parallax $\bar{p}$. The values of $\psi(b)$ are given in Table II.

TABLE II

Values of $\psi(b)$

hV	b	$\psi(b)$	hV	b	$\psi(b)$
0.0	0.0	1.000	1.0	0.50	1.446
0.1	0.005	1.005	1.1	0.605	1.529
0.2	0.02	1.020	1.2	0.72	1.616
0.3	0.045	1.043	1.3	0.845	1.706
0.4	0.08	1.078	1.4	0.98	1.800
0.5	0.125	1.121	1.5	1.125	1.896
0.6	0.18	1.172	1.6	1.28	1.994
0.7	0.245	1.231	1.7	1.445	2.094
0.8	0.32	1.297	1.8	1.62	2.197
0.9	0.405	1.369	1.9	1.805	2.300
1.0	0.50	1.446	2.0	2.00	2.404

14. We now derive the mean value of the transverse linear velocity, with respect to the Sun, of the stars forming a drift and scattered over the sky. We shall assume uniform distribution over the sphere; in practice, this will mean giving unit weight to the individual regions considered, irrespective of the number of stars in each area.

Since the number of stars per unit area is N , the number in the zone between λ and $\lambda + d\lambda$ (see fig. 1) is $2\pi N \sin \lambda d\lambda$. Let $\bar{\tau}_0$ denote the mean value of r taken over the whole sphere. Then by (58) we find

$$2\bar{\tau}_0 = \bar{T} \int_0^\pi \sin \lambda \psi(b) d\lambda. \quad (62)$$

Now V is the projection of the space-velocity U of the drift on the tangent plane at S , so that

$$V = U \sin \lambda.$$

Hence, from (54),

$$2b = (hU)^2 \sin^2 \lambda.$$

Let

$$2c = (hU)^2 \equiv \tau^2. \quad (63)$$

Then

$$b = c \sin^2 \lambda. \quad (64)$$

Thus, from (62) and (64),

$$\begin{aligned} \frac{\bar{\tau}_0}{\bar{T}} &= \int_0^{\pi/2} \sin \lambda \psi(b) d\lambda \\ &= \frac{1}{2\sqrt{c}} \int_0^c \frac{\psi(b)}{\sqrt{c-b}} db. \end{aligned} \quad (65)$$

Consider the integral on the right of (65); it is

$$\int_0^c \frac{db}{\sqrt{c-b}} \cdot e^{-b} \left\{ (1+2b) \mathfrak{f}_0(ib) + 2b \frac{d\mathfrak{f}_0(ib)}{db} \right\}.$$

Integrating by parts and using Bessel's equation,

$$\frac{d^2 \mathfrak{f}_0(y)}{dy^2} + \frac{1}{y} \frac{d\mathfrak{f}_0(y)}{dy} + \mathfrak{f}_0(y) = 0,$$

we obtain

$$\frac{\bar{\tau}_0}{\bar{T}} = 1 + \frac{1}{\sqrt{c}} \int_0^c \sqrt{c-b} e^{-b} \left\{ \mathfrak{f}_0(ib) + \frac{d\mathfrak{f}_0(ib)}{db} \right\} db$$

or

$$\frac{\bar{\tau}_0}{\bar{T}} = 1 + \frac{1}{\sqrt{c}} \int_0^c \sqrt{c-b} e^{-b} \{I_0(b) + I_1(b)\} db. \quad (66)$$

The expression on the right of (66) is a function of c or of τ , by (63), and we write (66) as

$$\bar{\tau}_0 = \bar{T} \chi(\tau). \quad (67)$$

For the stars in drift I and in drift II we shall have

$$\bar{\tau}_0' = \bar{T} \chi(\tau_1) \quad \text{and} \quad \bar{\tau}_0'' = \bar{T} \chi(\tau_2).$$

If $\bar{T}_0$ denotes the mean transverse velocity for all the stars of drifts I and II, and α and $(1-\alpha)$ denote as before the proportions of the numbers of stars in the two drifts, we obtain

$$\bar{T}_0 = f_2 \bar{T}, \quad (68)$$

where

$$f_2 = \alpha \chi(\tau_1) + (1-\alpha) \chi(\tau_2). \quad (69)$$

Assuming for statistical purposes that all the stars concerned have the same parallax, we have from (68)

$$\bar{\mu}_0 = f_2 \bar{\mu}, \quad (70)$$

where $\bar{\mu}_0$ denotes the mean of the observed proper motions taken over the sphere and $\bar{\mu}$ is the mean of the peculiar proper motions.

Using (48) and (68), formula (1) becomes

$$\bar{p} = 3 \cdot 016 \frac{f_1}{f_2} \cdot \frac{\bar{\mu}_0}{R_0}, \quad (71)$$

which is the proper expression of (2).

15. There remains the evaluation of the function $\chi(\tau)$ as given by the expression on the right of (66).

Numerical integration for the values $\tau = 0.8, 0.9, 1.0$ and $1.4, 1.5$ and 1.6 (within which ranges the actual values of τ_1 and τ_2 may be expected to occur) led to values of $\chi(\tau)$ identical with the corresponding values of $\phi(\tau)$ as given in Table I. This was an unexpected result, as the forms of the functions $\phi(\tau)$ and $\chi(\tau)$ hardly suggest this simple connection. The identity of the functions will be proved in the following section.

Taking $\phi(\tau) = \chi(\tau)$, it follows that in (71)

$$f_2 = f_1,$$

and we obtain the simple and convenient formula (2), namely,

$$\bar{p} = 3 \cdot 016 \frac{\bar{\mu}_0}{R_0},$$

which is the formula used by Edmondson in deriving statistical parallaxes. It may be added that Edmondson's derivation of this formula is equivalent to "proving" that $f_1 = f_2 = 1$, which is incorrect according to the preceding analysis, inasmuch as f_1 and f_2 are each numerically equal to 1.5 approximately when we use the usual values of the drift-constants.

16. *Proof of the identity $\phi(\tau) = \chi(\tau)$.*—By (44) and (46),

$$\begin{aligned} \phi(\tau) &= 1 + \frac{1}{\tau} \int_0^\tau (\tau - x)^2 e^{-x^2} dx \\ &= 1 + \frac{1}{\tau} \int_0^\tau (\tau - x)^2 \left\{ \sum_0^\infty (-1)^r \frac{x^{2r}}{r!} \right\} dx, \end{aligned}$$

the integrand being a uniformly convergent series.

Hence

$$\begin{aligned} \phi(\tau) &= 1 + \sum_0^\infty (-1)^r \frac{\tau^{2r+2}}{r!} \left\{ \frac{1}{2r+1} - \frac{2}{2r+2} + \frac{1}{2r+3} \right\} \\ &= 1 + 2 \sum_0^\infty (-1)^r \cdot \frac{\tau^{2r+2}}{(2r+1)(2r+2)(2r+3) r!} \end{aligned} \quad (72)$$

Again, from (55),

$$\begin{aligned} \bar{r}_0 &= \frac{1}{h\sqrt{\pi}} \int_0^{\pi/2} \left\{ (1+4b) - 4b \sin^2 \theta \right\} e^{-2b \sin^2 \theta} d\theta \\ &= \frac{1}{h\sqrt{\pi}} \int_0^{\pi/2} \left[1 + \sum_1^\infty (-1)^r (2b)^r \left\{ \frac{\sin^{2r} \theta}{r!} - \frac{2 \sin^{2r-2} \theta}{(r-1)!} + \frac{2 \sin^{2r} \theta}{(r-1)!} \right\} \right] d\theta. \end{aligned}$$

Now, by (24), $T = \sqrt{\pi}/2h$; hence, since the series is uniformly convergent,

$$\frac{\bar{r}_0}{T} = 1 + \sum_1^{\infty} (-1)^r (2b)^r \left\{ \frac{1+2r}{r} \cdot \frac{|2r|}{2^{2r}(|r|^2)} - \frac{2}{r-1} \cdot \frac{|2r-2|}{2^{2r-2}(|r-1|^2)} \right\};$$

or, using (58),

$$\psi(b) = 1 + 2 \sum_1^{\infty} (-1)^{r-1} \frac{b^r}{2^r} \frac{|2r-2|}{r-1(|r|^2)}$$

and this is a convergent series.

But, from (65),

$$\frac{\bar{r}_0}{T} = \chi(\tau) = \int_0^{\pi/2} \sin \lambda \psi(b) d\lambda,$$

where now $b = c \sin^2 \lambda$ and the integrand is a uniformly convergent series.

Hence

$$\begin{aligned} \chi(\tau) &= 1 + 2 \sum_1^{\infty} (-1)^{r-1} \frac{c^r |2r-2|}{2^r |r-1| (|r|^2)} \int_0^{\pi/2} \sin^{2r+1} \lambda d\lambda \\ &= 1 + 2 \sum_1^{\infty} (-1)^{r-1} \frac{(2c)^r}{(2r-1)(2r)(2r+1) |r-1|} \end{aligned}$$

Now $2c = \tau^2$, and we have

$$\chi(\tau) = 1 + 2 \sum_0^{\infty} (-1)^r \frac{\tau^{2r+2}}{(2r+1)(2r+2)(2r+3) |r|}$$

which is identical with (72). Thus

$$\phi(\tau) = \chi(\tau).$$

17. Summary.

1. Formulæ are derived giving

- (i) the mean observed radial velocity of stars scattered over the sky in terms of the mean peculiar radial velocity,
- (ii) the mean observed transverse linear velocity of stars in a small region of the sky in terms of the mean peculiar transverse velocity,
- (iii) the mean observed proper motion of stars scattered over the sky in terms of the mean peculiar proper motion.

2. The statistical formula

$$\bar{p} = 3.016 \frac{\bar{\mu}_0}{\bar{R}_0}$$

for deriving mean parallaxes is established.

Observatory, Cambridge :

1934 November 26.

SECOND ORDER TERMS IN DIFFERENTIAL REFRACTION IN THE MEASUREMENT OF PHOTOGRAPHIC PLATES.

H. Spencer Jones, Sc.D., F.R.S.

1. In a programme of observations, such as the programme of observations of Eros by the diurnal method for the determination of the solar parallax, in which photographs are obtained at large zenith distances, the second order terms in differential refraction become appreciable on astrographic plates covering $2^\circ \times 2^\circ$. For plates covering larger fields the second order terms have to be allowed for at smaller zenith distances. It is necessary that corrections for such terms should be applied before the scale and orientation of the plate are deduced from the measures of the reference stars. As is well known, the first order terms in differential refraction and aberration are included in the solution for the plate constants.

2. The following expressions for the second order terms in the differential refraction have been given :—

(a) Kapteyn * obtained :

$$\begin{aligned}\Delta x &= -kX\{(1+X^2)x^2 + 2XYxy + (1+Y^2)y^2\}, \\ \Delta y &= -kY\{(1+X^2)x^2 + 2XYxy + (1+Y^2)y^2\}.\end{aligned}$$

Expressions exactly equivalent to those given by Kapteyn were derived by another method by Baillaud,† and have been given also by Zurhellen,‡ Heckmann,§ König || and Vick.¶

(β) Turner ** obtained formulæ from which the following expressions (first given by Hinks ††) are easily derived :—

$$\begin{aligned}\Delta x &= -k\{X(2+X^2)x^2 + Y(1+2X^2)xy + X(1+Y^2)y^2\}, \\ \Delta y &= -k\{Y(1+X^2)x^2 + X(1+2Y^2)xy + Y(2+Y^2)y^2\}.\end{aligned}$$

Equivalent expressions were also obtained by Bemporad.‡‡

In the above formulæ the corrections are applicable to the measured co-ordinates to obtain co-ordinates free from the effects of the second order terms. x, y are the measured co-ordinates on the plate, the axis of y being the projection on the plate of the great circle from the centre of the plate

* *Bulletin du Com. Int. Perm. de la Carte du Ciel*, 3, 54, 1902.

† *Ibid.*, 3, 19, 1902 (*vide* p. 48).

‡ "Darlegung und Kritik der zur Reduktion photographischer Himmelsaufnahmen aufgestellten Formeln und Methoden" (Inaug. Diss., Bonn, 1904).

§ *Veroff. Göttingen*, Heft 30, 1932.

|| "Reduktion photographischer Himmelsaufnahmen," *Handb. d. Astrophysik*, 1, 529.

¶ *Ast. Nachr.*, 253, No. 6063, 1934.

** *M.N.*, 57, 133, 1897.

†† *Ibid.*, 63, 138, 1903.

‡‡ *Mem. Soc. Astr. It. (N.S.)*, 7, No. 1, 1933.

to the pole ; the axis of x is at right angles to the axis of y , in the direction of increasing right ascension. X , Y are the co-ordinates of the zenith on the plate and are given by

$$\begin{aligned} X &= \tan q = \sin \omega \tan \zeta, \\ Y &= \tan p = \cos \omega \tan \zeta, \end{aligned}$$

where ω denotes the parallactic angle at the centre of the plate and ζ is the true zenith distance of the centre of the plate. k is the coefficient of refraction, the effect on the zenith distance being given by

$$\Delta \zeta = -k \tan \zeta.$$

k is variable with ζ and can be approximately expressed in the form :

$$k = k_0 + k_1 \tan^2 \zeta.$$

In the above formulæ and in what follows k_1 is assumed to be zero, *i.e.* the variation of the refraction constant with zenith distance is neglected.

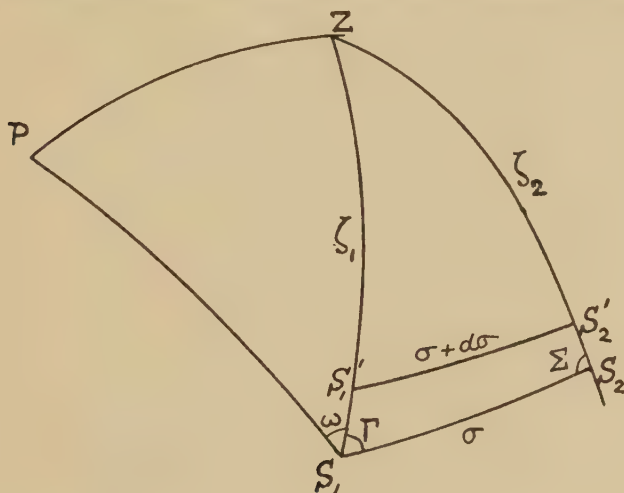


FIG. 1.

3. The purpose of the present note is to call attention to the discordance between the above expressions, to establish the correctness of the formulæ given by Turner, and to show where the error is involved in the investigation of Kapteyn. Turner's method is simple and elegant, as it deals *ab initio* with the co-ordinates on the photographic plate. The methods of Kapteyn and those who have followed him are long and unnecessarily involved. Kapteyn derived first the effect of differential refraction on the distance apart of two stars and on the position angle of the arc of a great circle from the one to the other. He then deduced the differential refraction effects on the rectangular co-ordinates on the photographic plate.

4. The effect of differential refraction on distance and position angle may be derived as follows :—

In fig. 1 Z denotes the zenith, P the pole, S_1 , S_2 the true positions of

two stars at zenith distances ζ_1, ζ_2 . The effect of refraction is to displace these towards the zenith, to the positions S_1', S_2' , where

$$S_1 S_1' = k \tan \zeta_1, \quad S_2 S_2' = k \tan \zeta_2.$$

The true distance $S_1 S_2$ is denoted by σ , the apparent distance $S_1' S_2'$ by $\sigma + d\sigma$. The angle $ZS_1 S_2$ is denoted by Γ , the angle $ZS_1' S_2'$ by $\Gamma + d\Gamma$, the angle $S_1 Z S_2$ by a .

From

$$\cos \sigma = \cos \zeta_1 \cos \zeta_2 + \sin \zeta_1 \sin \zeta_2 \cos a$$

by differentiating and using

$$d\zeta_1 = -k \tan \zeta_1, \quad d\zeta_2 = -k \tan \zeta_2,$$

we obtain on reduction

$$\sin \sigma d\sigma = -k(\cos \zeta_2 \sec \zeta_1 + \sec \zeta_2 \cos \zeta_1 - 2 \cos \sigma)$$

Eliminating ζ_2 by using

$$\cos \zeta_2 = \cos \sigma \cos \zeta_1 + \sin \sigma \sin \zeta_1 \cos \Gamma,$$

we obtain

$$d\sigma = -k \cdot \frac{1 + \tan^2 \zeta_1 \cos^2 \Gamma}{1 + \tan \sigma \tan \zeta_1 \cos \Gamma} \tan \sigma \quad (1)$$

Also by differentiating logarithmically

$$\sin \Gamma = \sin \zeta_2 \sin a / \sin \sigma$$

we obtain

$$\cot \Gamma d\Gamma = -k - \cot \sigma d\sigma$$

or, using (1),

$$d\Gamma = k \sin \Gamma \tan \zeta_1 \frac{\cos \Gamma \tan \zeta_1 - \tan \sigma}{1 + \tan \sigma \tan \zeta_1 \cos \Gamma} \quad (2)$$

From the triangle $PS_1 S_1'$ we have at once, if angle $PS_1 Z = \omega$,

$$d\omega = k \sin \omega \tan \delta \tan \zeta_1 \quad (3)$$

The formulæ (1), (2), (3) agree with those derived by Kapteyn. $d\sigma$ gives the effect of differential refraction on distance. $d(\Gamma + \omega)$ is the total differential refraction in position angle. In projection on the tangent plane (the photographic plate) at S_1 or S_1' , the effect of $d\omega$ is equivalent to a rotation of axes. Such a rotation is automatically taken account of in the reduction of the plate. $d\omega$ can therefore be left out of consideration.

5. Kapteyn then proceeds as follows. If S_1 is the centre of the photographic plate and the axes of x, y on the plate are taken in the direction of right ascension and declination respectively, the co-ordinates x, y of S_2 on the plate are :

$$\begin{aligned} x &= \tan \sigma \sin (\Gamma + \omega) \\ y &= \tan \sigma \cos (\Gamma + \omega) \end{aligned}$$

On differentiating, and neglecting the change in ω ,

$$\begin{aligned} dx &= x \cot \sigma (1 + \tan^2 \sigma) d\sigma + y d\Gamma \\ dy &= y \cot \sigma (1 + \tan^2 \sigma) d\sigma - x d\Gamma \end{aligned}$$

Kapteyn substitutes the values of $d\sigma$, $d\Gamma$ from (1), (2) in these expressions, and obtains expressions for the differential refraction in rectangular co-ordinates on the plate, the second order terms of which are equivalent to those given in § 2 (a), as may easily be verified by using the relations :

$$\left. \begin{aligned} X^2 + Y^2 &= \tan^2 \zeta_1, & x^2 + y^2 &= \tan^2 \sigma \\ Xx + Yy &= \tan \sigma \tan \zeta_1 \cos \Gamma \\ Yx - Xy &= \tan \sigma \tan \zeta_1 \sin \Gamma \end{aligned} \right\} \quad (4)$$

6. The error in this derivation arises in the following way : If the photographic plate is considered as being tangential to the celestial sphere at S_1 ,

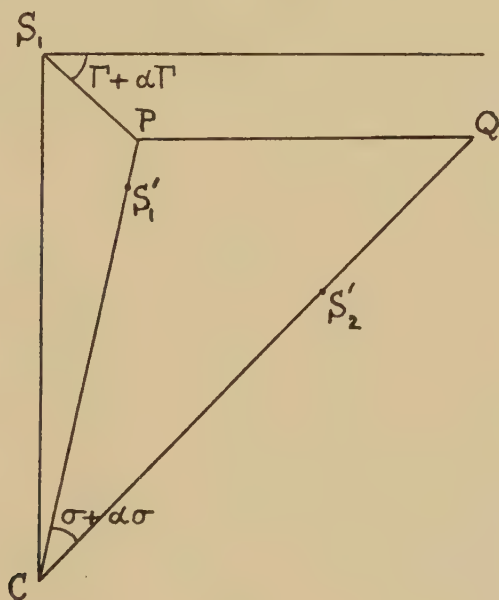


FIG. 2.

the projection of S_1S_2 on the plate is $\tan \sigma$. It is implicitly assumed by Kapteyn that the projection of $S'_1S'_2(\sigma + d\sigma)$ is $\tan(\sigma + d\sigma)$. This is not correct to the second order. Alternatively, if the photographic plate is considered as being at S'_1 , the projection of $S'_1S'_2$ on the plate is $\tan(\sigma + d\sigma)$, but the projection of S_1S_2 is no longer $\tan \sigma$.

7. The error which is thus introduced can be determined without difficulty. In fig. 2, C is the centre of projection, S_1 the centre of the plate, P, Q the projections of S'_1, S'_2 on the plate. Then $S_1P = S_1S'_1 = k \tan \zeta_1$, correct to the second order. Also the angle PCQ is $\sigma + d\sigma = \sigma'$ (say).

If we take co-ordinate axes through C , with CS_1 as axis of z and the axis of x parallel to PQ , then since the radius of the sphere is unity, the

co-ordinates of P are $(k \tan \zeta_1 \cos \Gamma', k \tan \zeta_1 \sin \Gamma', 1)$, where we have put $\Gamma + d\Gamma = \Gamma'$.

The co-ordinates of Q are $(\xi, k \tan \zeta_1 \sin \Gamma', 1)$, where ξ is required. We have, since angle PCQ is σ' ,

$$\cos \sigma' = (1 + k\xi \tan \zeta_1 \cos \Gamma') / \sqrt{1 + \xi^2}$$

whence

$$\xi = \tan \sigma' + k \sec^2 \sigma' \tan \zeta_1 \cos \Gamma'$$

Hence

$$\begin{aligned} PQ &= \xi - k \tan \zeta_1 \cos \Gamma' \\ &= \tan \sigma' + k \tan^2 \sigma' \tan \zeta_1 \cos \Gamma' \\ &= \tan (\sigma' + \delta\sigma), \end{aligned}$$

where

$$\begin{aligned} \delta\sigma &= k \sin^2 \sigma' \tan \zeta_1 \cos \Gamma' \\ &= k \sin^2 \sigma \tan \zeta_1 \cos \Gamma \end{aligned} \quad (5)$$

Kapteyn's derivation assumes that $PQ = \tan (\sigma + d\sigma) = \tan \sigma'$. Hence in changing to co-ordinates on the plate, the value of $d\sigma$ given in (1) needs correction by the amount $\delta\sigma$ given in (5).

8. The appropriate value of $d\sigma$ to use for the projection of $S_1'S_2'$ on the plate can be easily deduced directly. In fig. 3, S_1 is the centre of the

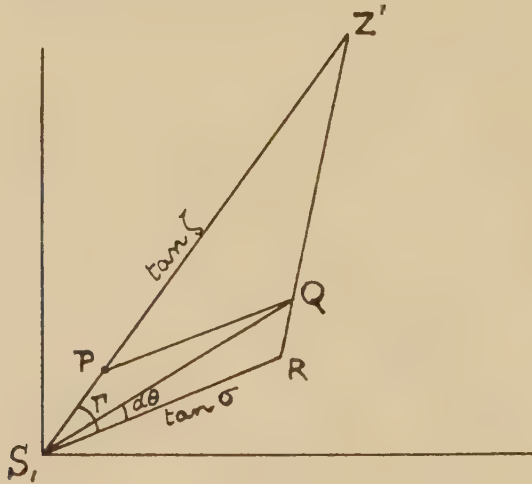


FIG. 3.

plate; P, Q are the projections of S_1', S_2' respectively. R is the projection of S_2 ; Z' is the projection of the zenith. Then $Z'PS_1$ lie on a straight line and $Z'QR$ lie on a straight line, both being projections of a great circle. Also $S_1R = \tan \sigma$; $S_1Z' = \tan \zeta_1$, and we write $PQ = \tan (\sigma + d\sigma)$. We require to determine the value of $d\sigma$.

The angle QS_1R in fig. 3 is equal to the angle $S_2'S_1S_2$ in fig. 1, since the angle between two great circles is equal to the angle between their projections on the tangent plane at the point of intersection. Denoting this angle by $d\theta$ and angle S_1S_2Z by Σ , we have from triangle $S_1S_2S_2'$:

$$\sin \sigma d\theta = k \tan \zeta_2 \cdot \sin \Sigma$$

and from triangle ZS_1S_2 ,

$$\sin \Sigma \sin \zeta_2 = \sin \Gamma \sin \zeta_1$$

whence

$$\begin{aligned} d\theta &= k \sin \zeta_1 \sin \Gamma / \sin \sigma \cos \zeta_2 \\ &= \frac{k \tan \zeta_1 \sin \Gamma}{\sin \sigma \cos \sigma (1 + \tan \sigma \tan \zeta_1 \cos \Gamma)} \end{aligned}$$

Also, in fig. 3, from the triangles S_1QR , S_1RZ' ,

$$QR = \frac{\tan \sigma d\theta}{\tan \zeta_1 \sin \Gamma} RZ',$$

or

$$d(RZ') = - \frac{k \sec^2 \sigma}{1 + \tan \sigma \tan \zeta_1 \cos \Gamma} RZ'$$

Then, from triangle S_1RZ' ,

$$\tan^2 \sigma = \tan^2 \zeta_1 + RZ'^2 - 2 \tan \zeta_1 \cdot RZ' \cos S_1Z'R$$

by differentiating, remembering that $d(\tan \zeta_1) = -k \tan \zeta_1$, and substituting $RZ' \cos S_1Z'R = \tan \zeta_1 - \tan \sigma \cos \Gamma$, we obtain on reduction,

$$d\sigma = -k \tan \zeta_1 \cos \Gamma \cos^2 \sigma + k \frac{\tan \zeta_1 \cos \Gamma - \tan \sigma}{1 + \tan \sigma \tan \zeta_1 \cos \Gamma} \quad (6)$$

It is easily seen that the difference between the value of $d\sigma$ given in (6), and the value derived above in (1) (which is the correct expression for the effect of differential refraction on an arc), is exactly $k \sin^2 \sigma \tan \zeta \cos \Gamma$, in agreement with the difference obtained in (5).

9. With the value of $d\sigma$ from (6) and the value of $d\Gamma$ from (2), the differential formulæ for dx , dy as given in § 5 are valid. Using (4), we obtain

$$\begin{aligned} (x^2 + y^2) d\sigma \cot \sigma (1 + \tan^2 \sigma) &= -k(Xx + Yy) + k \frac{x(X - x) + y(Y - y)}{1 + xX + yY} (1 + x^2 + y^2) \\ (x^2 + y^2) d\Gamma &= +k(Yx - Xy) + k \frac{y(X - x) - x(Y - y)}{1 + xX + yY} (1 + x^2 + y^2) \end{aligned}$$

whence it follows that

$$\begin{aligned} dx &= -kX + k(X - x) \frac{1 + x^2 + y^2}{1 + xX + yY} \\ dy &= -kY + k(Y - y) \frac{1 + x^2 + y^2}{1 + xX + yY} \end{aligned}$$

These expressions are those obtained by Turner (*loc. cit.*), and lead at once to the second order terms given in § 2 (β).

10. Kapteyn's method introduces the unnecessary complication of dealing with the refraction in distance and position angle between two stars, both of which are affected by refraction. One of these stars is assumed to be at the centre of the plate, which is a definite point on the celestial sphere and is not affected by refraction.

In fig. 4, suppose O is the centre of the plate, the arc ZO being ζ_0 , and that S is a star, at zenith distance ζ , which is lifted by refraction to S' , so that $SS' = k \tan \zeta$.

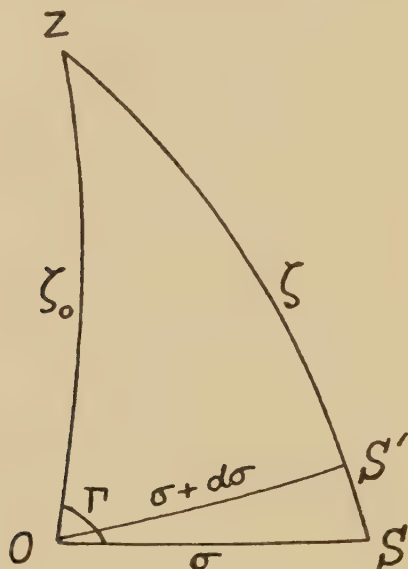


FIG. 4.

Writing $OS = \sigma$, $OS' = \sigma + d\sigma$, $ZOS = \Gamma$, $ZOS' = \Gamma + d\Gamma$, we readily find by differentiating

$$\cos \sigma = \cos \zeta_0 \cos \zeta + \sin \zeta_0 \sin \zeta \cos a$$

that

$$d\sigma = k \cdot \frac{\tan \zeta_0 \cos \Gamma - \tan \sigma}{1 + \tan \sigma \tan \zeta_0 \cos \Gamma} \quad (7)$$

and from

$$\sin \Gamma \sin \sigma = \sin \zeta \sin a,$$

we have

$$\cot \Gamma d\Gamma = -k - \cot \sigma d\sigma,$$

whence

$$d\Gamma = \frac{-k}{\sin \sigma \cos \sigma} \frac{\tan \zeta_0 \sin \Gamma}{(1 + \tan \sigma \tan \zeta_0 \cos \Gamma)} \quad (8)$$

Using (4), we obtain

$$\begin{aligned} \tan \sigma d\sigma &= k \cdot \frac{Xx + Yy - (x^2 + y^2)}{1 + Xx + Yy}, \\ \tan^2 \sigma d\Gamma &= -k \cdot \frac{(1 + x^2 + y^2)(Yx - Xy)}{1 + Xx + Yy}. \end{aligned}$$

The formulæ for dx , dy given in § 5 are correct, where x , y denote rectangular co-ordinates on a plate centred at O , because the projection of the arcs σ and $(\sigma + d\sigma)$ on the plate are now correctly $\tan \sigma$, $\tan (\sigma + d\sigma)$.

We at once obtain

$$dx = k(X - x) \frac{1 + x^2 + y^2}{1 + xX + yY},$$

$$dy = k(Y - y) \frac{1 + x^2 + y^2}{1 + xX + yY}.$$

These agree with the expressions given in § 9, except for the terms $-kX$, $-kY$ in dx , dy respectively. In the present case when x , y are zero,

$$dx = kX, \quad dy = kY,$$

these quantities expressing the displacement due to refraction of a star whose true right ascension and declination are those of the centre of the plate.

These formulæ therefore necessarily lead again to Turner's formulæ for the second order terms in differential refraction. The method of derivation is not so elegant as the method used by Turner, but it is of interest to compare it with the method used by Kapteyn and outlined above.

11. The conclusion is that the correct formulæ for the second order terms in differential refraction are those given by Turner (*vide* § 2 (β) above), and that the formulæ given by Kapteyn and others are incorrect. This conclusion is contrary to the conclusion of Zurhellen (*loc. cit.*).

THE CHARACTERISTIC CURVES OF SOME VARIABLE STARS.

I. A. Getting.

(Communicated by E. A. Milne, F.R.S.)

1. The purpose of this paper is the application of available light and radial velocity curves to find the characteristic curves of different types of variable stars by Milne's method.* Briefly, Milne's method consists of plotting values of the star's radius against simultaneous values of its bolometric luminosity, giving the (r, L) diagram, under the assumption that the star's variability is due to pulsation. Such a diagram with time or phase markings gives the complete history of a variable; for not only does it give phase relations, but also the effective temperatures at all times—in short, all of the observable physical quantities. As will be evident later, this diagram also fixes the type of the star and therefore suggests a simple method of classification for variables. A list of the stars for which data were available in the literature and for which characteristic curves are given later in this paper is given in Table I. In each case the first reference is the source of the luminosity curve together with the epochs covered by the observations; and the second, the source of the radial velocity curve also with the epochs covered.

* Milne, *M.N.*, 94, 418, 1934.

TABLE I

η Aquilæ 19 ^h 47 ^m .4 +0° 45'	Petit and Nicholson, <i>Ap. J.</i> , 78, 342, 1933. 1922 June 27 to 1926 July 6. Jacobsen, <i>L.O.B.</i> , 12, 138, 1926. 1923 July 2 to 1925 December 15.
W Sagittarii 17 ^h 58 ^m .6 -29° 35'	Robinson, <i>An. H.C.O.</i> , 90, 36, 1933. 1899 to 1930. Jacobsen, <i>L.O.B.</i> , 12, 138, 1926. 1923 June 12 to 1925 July 20.
δ Cephei 22 ^h 25 ^m .4 +57° 54'	Petit and Nicholson, <i>Ap. J.</i> , 78, 342, 1933. 1922 July 24 to 1926 July 16. Jacobsen, <i>L.O.B.</i> , 12, 138, 1926. 1923 July 7 to 1923 December 1.
RR Lyræ 19 ^h 22 ^m .3 +42° 36'	Robinson, <i>An. H.C.O.</i> , 90, 27, 1933. 1899 to 1930. Sanford, <i>Ap. J.</i> , 67, 319, 1928. Mostly during 1926 and 1927.
α Ceti 2 ^h 14 ^m .3 +3° 26'	Petit and Nicholson, <i>Ap. J.</i> , 78, 336, 1933. 1922 July 27 to 1924 August 30. Joy, <i>Ap. J.</i> , 63, 281, 1926. 1916 November to 1926 January.
SX Herculis 16 ^h 3 ^m .3 +25° 11'	Gerasamovič, <i>H.C. Bul.</i> , 869, 11, 1929. 1900 to 1929. Joy, <i>Ap. J.</i> , 75, 127, 1932. 1926 August to 1931 July.
R Virginis 12 ^h 33 ^m .4 +7° 32'	Merrill and Joy, <i>Ap. J.</i> , 69, 379, 1929. <i>Ibid.</i> 1922 February to 1928 July.
AC Herculis 19 ^h 26 ^m .0 21° 17.6'	Leiner, <i>A.N.</i> , 221, 247, 1923. Before 1923. Sanford, <i>Ap. J.</i> , 73, 364, 1931. 1924 October to 1930 September.
U Monocerotis 7 ^h 26 ^m .0 +9° 34'	Loreta, <i>Bul. de l'Association Française d'Observateurs d'Etoiles Variable</i> , 2, 97, 1933. 1932 January 7 to 1932 April 14. Sanford, <i>Ap. J.</i> , 77, 120, 1933. 1927 to 1933.

2. *Construction of the (r, L) Diagram.*—For a complete discussion of the method the reader is referred to Milne's paper. However, for the sake of clarity a summary is given.

(a) The radial velocity of the centre of gravity of the variable star is found by taking the time mean of the radial velocity curve as determined from absorption lines.

$$V_m = \frac{1}{T} \int_0^T V_r dt.$$

(b) The pulsation velocity v_r is then found by multiplying the difference between V_r and V_m by a factor a to correct for components of velocity not in the line of sight and for darkening at the limb ;

$$v_r = a(V_r - V_m).$$

As all pulsating variables are giants, the theory of radiative equilibrium should be particularly applicable to their photospheres. The specific intensity is given * by $\frac{I_{\theta}}{I_0} = 1 - \alpha + \alpha \cos \theta$, where θ is the angle between the line of sight and the normal to the surface of the star, and α assumes the value $3/5$ for radiative equilibrium, 1 for Lambert's law and 0 for equal intensity in all directions.

The average value of $v_r \cos \theta$ in the line of sight is

$$\bar{v}_r = \frac{\iint (v_r \cos \theta) I(\theta) \cos \theta d\omega}{\iint I(\theta) \cos \theta d\omega}.$$

Substituting for $d\omega$ in terms of $d\theta$,

$$\bar{v}_r = \frac{2\pi v_r \int_0^{\pi/2} (1 - \alpha + \alpha \cos \theta) \cos^2 \theta \sin \theta d\theta}{2\pi \int_0^{\pi/2} (1 - \alpha + \alpha \cos \theta) \cos \theta \sin \theta d\theta},$$

and on integrating,

$$\bar{v}_r = v_r \left[\frac{4 - \alpha}{6 - 2\alpha} \right],$$

$$v_r = \frac{2}{1} \frac{4}{8} \bar{v}_r ; \quad \alpha = 0,$$

$$v_r = \frac{2}{1} \frac{4}{7} \bar{v}_r ; \quad \alpha = \frac{3}{5},$$

$$v_r = \frac{2}{1} \frac{4}{8} \bar{v}_r ; \quad \alpha = 1.$$

It is at once apparent that the dependence of a on the function $I(\theta)$ is very insensitive. The value $24/17$, representing approximately radiative equilibrium, may be taken as being the most probable value of a —in any case the error cannot be great.†

(c) The pulsation velocity curve is integrated graphically from the time of minimum radius t_0 to definite phase intervals t_p to get the increase over the minimum radius at the time t_p ,

$$(r - r_{\min}) = \int_{t_0}^{t_p} v_r dt.$$

(d) The value of $r_{\min}$ is estimated and added to $(r - r_{\min})$. In the actual plots below, the method for deriving the value of $r_{\min}$ is in each case noted.

* E. A. Milne, *Handbuch der Astrophysik*, 3, 118, 1930.

† In a more refined method, each absorption line should be corrected for its particular darkening as the darkening is a function of wave-length. Such refinement in the present problem would be a luxury.

(e) The logarithms of the radius are plotted against $-m_b$ the bolometric magnitude, by definition proportional to $\log_{10} L$, as ordinate.

(f) The effective temperature is determined at some definite phase from inspection of available data and a grid of temperature lines is constructed from the relation,

$$L_b = 4\pi r^2 \sigma T_e^4,$$

or

$$-m_b = 5 \log_{10} r + 10 \log_{10} T_e + C. \quad (1)$$

Loci of constant temperature will be straight lines of slope $1/5$.

3. *Derivation of the Bolometric Luminosity.*—If a bolometric light curve of a variable star is available, the appropriate relation between m_b and r is (1). Unfortunately the correction to bolometric magnitudes requires not only a knowledge of the distribution of intensity as a function of wave-length, but also the effective temperatures at all times. If the effective temperature can be established at one phase, then Milne's method permits the finding of the bolometric luminosity for every other point in the cycle. The advantage is that whereas the colour temperature, the temperature normally used for correcting to bolometric luminosity, can fluctuate even to both sides of the effective temperature, the deviations of the bolometric corrections are small for deviations of the distribution curve from that of a black body. Hence the bolometric curve based on the (r, L) diagram—that is, on effective temperatures—is likely to be more reliable than one based on colour temperatures. The bolometric luminosity is obtained as follows:—

The visual magnitude curve is corrected to a bolometric curve * as if at all times the effective temperature were constant and had the value fixed at a particular phase by the best means available. With these fictitious bolometric luminosities the (r, L) diagram is constructed. Temperatures are then read off for each time marking along the curve from the temperature grid so fixed as to go through the chosen phase of fixed effective temperature. With these temperatures a revision can be made of the bolometric luminosities at the various phase intervals. A second approximation from the new (r, L) diagram is made in the same manner. It has been found that three approximations were sufficient to give repeated values of the bolometric magnitudes in most cases. The number of successive approximations never exceeded five.

4. *The Visual and Photographic Magnitude Plots.*—Instead of using bolometric magnitudes, it is possible to construct an (r, L) diagram immediately from observational luminosity curves in terms of visual or photographic magnitudes. While these curves do not lend themselves as simply to theoretical discussions,† they have the advantage of introducing fewer approximations. It is no longer necessary to correct for atmospheric absorption, but it is still necessary to assume black-body distribution to evaluate brightness temperatures for one point on each plot for fixing the temperature grids from observational temperatures. Due to the narrowness of the response-curves

* Eddington, *I.C.S.*, 138, 1930.

† See Milne's original article, *M.N.*, 94, 418, 1934.

of the eye and of photographic materials, radiation of Planck's distribution may be considered as being all of one effective wave-length, the relation connecting the brightness temperature, the effective wave-length and the brightness being

$$\mathcal{J}_\lambda = \frac{C\lambda^{-5}}{10^{\kappa/\lambda T_b} - 1}. \quad (2)$$

The response-curve of the eye varies with the observer and the level of illumination. Likewise the response-curve varies with emulsion, development and other factors for the photographic plate. For lack of any more exact information, Brill's* values of effective wave-length of 4250 Å. for the photographic magnitudes and 5290 Å. for the visual magnitudes have been used. For temperatures occurring at the surfaces of stars the second term of the denominator of equation (2) is negligible as compared with the first, and Planck's law reduces to

$$-m_p = -\frac{36700}{T_b} + 5 \log r - D_p, \quad (3)$$

$$-m_v = -\frac{29500}{T_b} + 5 \log r - D_v, \quad (4)$$

where D_p and D_v are constants.

Whereas for the plot of bolometric magnitudes the spacing of the isotherms depends logarithmically upon T_e , for the visual or photographic magnitudes the spacing is hyperbolic. Consequently the characteristic curves of the same star in the two types of plots will be somewhat deformed relative to each other. It should be possible to reduce the radiometric magnitude plot to the visual by use of the heat index, or from the photographic to the visual by use of the colour index, for the temperatures for all points are fixed by the temperature grid systems. Deviations between predicted curves and observed curves are then deviations from black-body conditions. For η Aquilæ, curves are given below showing a deviation of this kind.

5. *Errors Introduced in the Choice of Minimum Radius.*—The estimate of the minimum radius is in all cases difficult and uncertain. It has, however, little effect upon the shape of the curve, for if $\log r$ is expanded,

$$\begin{aligned} \log_{10} r &= 0.4343 \log_e r = 0.4343 \log_e(r_{\min} + \Delta r) \\ &= 0.4343 \log_e r_{\min} + 0.4343 \left[\frac{\Delta r}{r_{\min}} - \frac{1}{2} \left(\frac{\Delta r}{r_{\min}} \right)^2 + \dots \right] \\ &= \log_{10} r_{\min} + 0.4343 \left[\frac{\Delta r}{r_{\min}} - \frac{1}{2} \left(\frac{\Delta r}{r_{\min}} \right)^2 + \dots \right], \end{aligned}$$

and since $\frac{\Delta r}{r_{\min}}$ is of the order of .05,† terms of $\left(\frac{\Delta r}{r_{\min}} \right)^2$ and higher can be

* Russell, Dugan, Stewart, *Astronomy*, 2, 733, 1927.

† Eddington, *I.C.S.*, 185, 1930.

neglected, so that the effects of an error in $r_{\min}$ are a change in the additive constant and a multiplying factor which changes the scale of the plot.

On the other hand, the actual variation of the temperature does depend upon the minimum radius chosen. For the visual or photographic plot we may write

$$m = \frac{d}{T_b} - 5 \log_{10} r + D.$$

Writing now $r(t) = r_{\min} + \Delta r$, we have

$$m = \frac{d}{T_b} - 5 \log_{10} (r_{\min} + \Delta r) + D;$$

but also, putting $\Delta r = 0$, $T_b = (T_b)r_{\min}$, $m = (m)r_{\min}$,

$$(m)r_{\min} = \frac{d}{(T_b)r_{\min}} - 5 \log_{10} r_{\min} + D.$$

Subtracting the two equations, we have

$$m - (m)r_{\min} = d \left(\frac{1}{T_b} - \frac{1}{(T_b)r_{\min}} \right) - 5 \log_{10} \left(\frac{r_{\min} + \Delta r}{r_{\min}} \right).$$

Here $(m - (m)r_{\min})$ and Δr are fixed for any definite phase, so that

$\left(\frac{1}{T_b} - \frac{1}{(T_b)r_{\min}} \right)$ depends upon the choice of $r_{\min}$.

The sensitiveness of the range in temperature can be easily calculated. Using the subscript m for values when T is maximum, and 0 when T is minimum, we have

$$m_m - m_0 = d \left(\frac{1}{T_{bm}} - \frac{1}{T_{b0}} \right) - 5 \log_{10} \left(\frac{r_{\min} + \Delta r_m}{r_{\min} + \Delta r_0} \right),$$

$$\frac{T_m - T_0}{T_0 T_m} = -\frac{1}{d} \left\{ m_m - m_0 + 5 \log_{10} \left(\frac{r_{\min} + \Delta r_m}{r_{\min} + \Delta r_0} \right) \right\}.$$

Writing ΔT for the range, and $T_0 T_m$ as $\bar{T}^2$, $\bar{T}$ being now approximately the mean value, we have

$$\frac{\Delta T}{\bar{T}^2} = -\frac{1}{d} \left\{ m_m - m_0 + 5 \log_{10} \left(1 + \frac{\Delta r_m - \Delta r_0}{r_{\min}} \right) \right\}.$$

Expanding the logarithm, and again neglecting higher terms in $\frac{\Delta r_m - \Delta r_0}{r_{\min}}$,

$$\frac{\Delta T}{\bar{T}^2} = -\frac{1}{d} \left\{ m_m - m_0 + 5(0.4343) \left(\frac{\Delta r_m - \Delta r_0}{r_{\min}} \right) \right\}.$$

Finally, differentiating

$$\delta(\Delta T) = 5(0.4343) \frac{\bar{T}^2}{d} \left(\frac{\Delta r_m - \Delta r_0}{r_{\min}^2} \right) \delta r_{\min}. \quad (5)$$

As $\Delta r_m < \Delta r_0$ (see plots below), an increase in $\delta r_{\min}$ means a decrease in the range of temperature variation. Actual values for W Sagittarii and RR Lyræ are given in the discussion of the respective stars below.

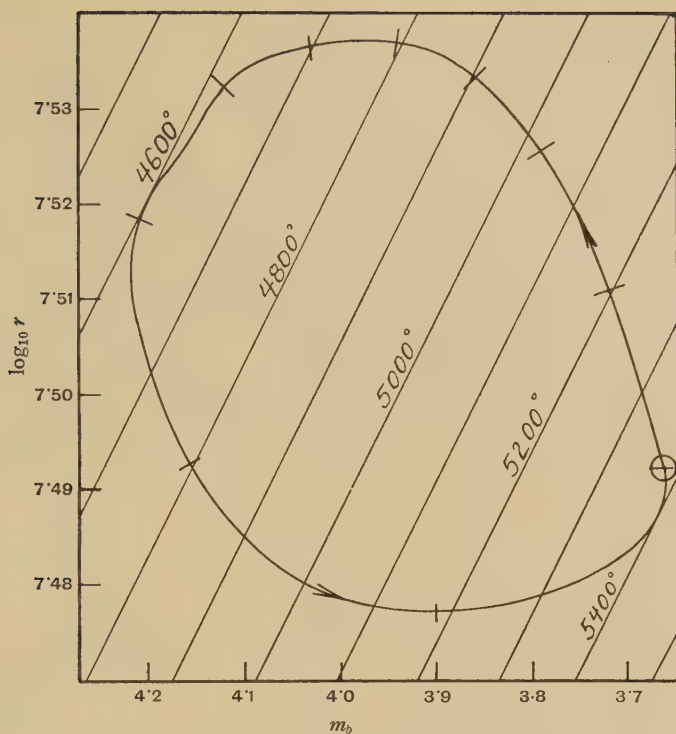


FIG. 1a.—Characteristic curve for η Aquilæ in terms of effective temperatures and bolometric magnitudes as derived from the radiometric magnitudes by successive approximations. Phase markings are in tenths of phase measured from maximum light.

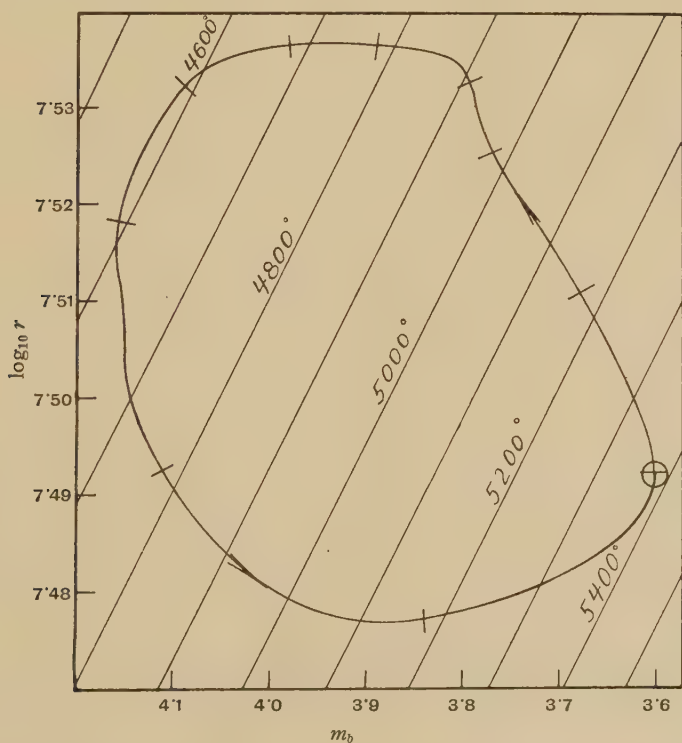


FIG. 1b.—Characteristic curve for η Aquilæ in terms of effective temperatures and bolometric magnitudes derived from visual magnitudes by successive approximations.

6. *Cepheids*.—Superimposed curves * of the radial velocities and magnitudes of twenty Cepheids have their maximum velocity of approach nearly coincident with maximum luminosity. This at once fixes the general rule that all Cepheids (including Cluster-type variables) have their representative curves traced in the anti-clockwise direction. It is interesting to study more in detail actual plots of certain Cepheids. η Aquilæ will be given first in detail, and then δ Cephei, W Sagittarii and RR Lyræ.

η Aquilæ, a Cepheid variable of period 7.1765 days, has been studied extensively by a great variety of methods. It is possible therefore to plot the characteristic bolometric curve as derived from both visual and radiometric luminosity curves. The two curves are shown in figures 1a and 1b.

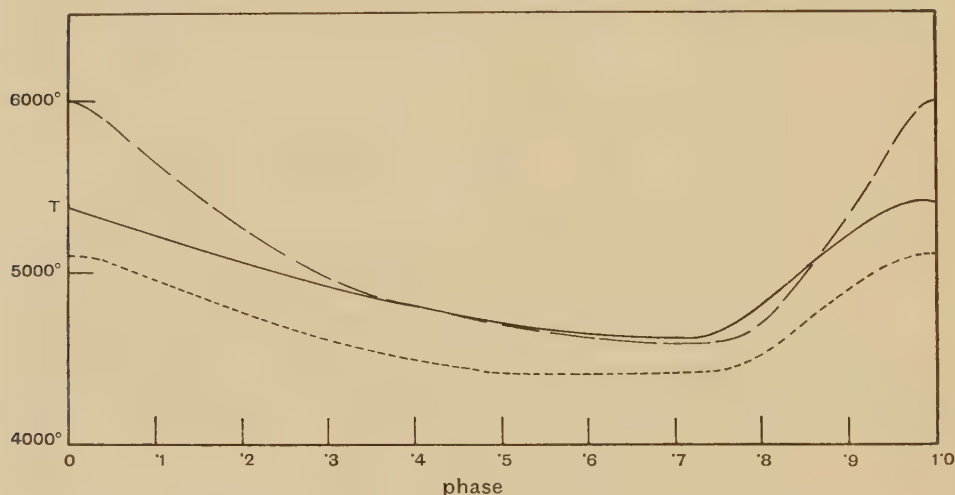


FIG. 2.—Temperature variation in η Aquilæ. The full line is the variation in effective temperature from the characteristic plot 1a. The dashed line is the spectral type temperatures (after Whipple). The dotted line is the colour temperature measured radiometrically (after Petit and Nicholson).

The curves appear shifted relative to each other 0.04 magnitude, a trivial amount. The bolometric curve derived from visual magnitudes was not corrected for atmospheric absorption. In general the agreement is exceedingly good, the small element of disagreement showing departure from black-body conditions.

The minimum radius was estimated from a parallax † of 0".005 and an angular diameter of 0".0021 as estimated by Petit and Nicholson, ‡ as well as from Shapley's period-luminosity curve. The effective temperature was taken arbitrarily as 5380°, being the average of three determinations by different methods. § In Table II, column A is the fraction of the total period from maximum light; column B, radiometric magnitude; column

* Robinson and Hoffleit, *H.C.O. Bulletin*, 888, 312, 1932.

† Shapley, *Star Clusters*, 139, 1930.

‡ Petit and Nicholson, *Ap. J.*, 78, 320, 1933.

§ Hopmann, *A.N.*, 222, 1, 1924; Petit and Nicholson, *Ap. J.*, 78, 320, 1933; Whipple, *L.O.B.*, 16, 1, 1932.

C, visual magnitude; column D, $r(t) - r_{\min}$; column E, $r(t)$ in km.; and column F, $\log_{10} r(t)$. In Table III, A is as in Table II, B is the bolometric magnitude derived from successive approximations of the radiometric magnitudes -0.4 magnitude; C is the bolometric magnitudes derived from five successive approximations of the visual magnitudes; and D, the difference of columns C and B. Fig. 2 shows the temperature derived from plot 1a as compared with the temperatures of Whipple (spectrophotometric) and Petit and Nicholson (radiometric colour temperatures).

TABLE II

A	B	C	D	E	F
0.00	3.20	3.70	10.51×10^5	310.5×10^5	7.4921
.05	3.21	3.75	17.65	317.7	.5020
.10	3.24	3.83	24.15	324.2	.5108
.15	3.27	3.89	30.64	330.6	.5193
.20	3.30	3.95	35.44	335.4	.5256
.25	3.33	3.98	38.97	339.0	.5302
.30	3.37	3.99	41.23	341.2	.5330
.35	3.40	4.02	43.35	343.4	.5358
.40	3.44	4.12	44.21	344.2	.5368
.45	3.48	4.19	44.34	344.4	.5370
.50	3.53	4.25	44.30	344.3	.5369
.55	3.57	4.32	43.35	343.4	.5358
.60	3.62	4.39	40.81	340.8	.5325
.65	3.66	4.44	35.60	335.6	.5258
.70	3.71	4.46	29.93	329.9	.5184
.75	3.71	4.42	21.32	321.3	.5069
.80	3.66	4.34	11.01	311.0	.4928
.85	3.56	3.20	2.69	302.7	.4810
.90	3.42	3.98	0.00	300.0	.4771
.95	3.26	3.80	3.42	303.4	.4820

TABLE III

A	B	C	D
0.00	3.62	3.60	+ .02
.05	3.64	3.63	+ .01
.10	3.68	3.68	+ .00
.15	3.71	3.73	- .02
.20	3.75	3.77	- .02
.25	3.78	3.78	+ .00
.30	3.82	3.79	+ .03
.35	3.85	3.82	+ .03
.40	3.90	3.89	+ .01
.45	3.94	3.94	+ .00
.50	3.99	3.98	+ .01
.55	4.03	4.04	- .01
.60	4.08	4.09	- .01
.65	4.12	4.13	- .01
.70	4.17	4.16	+ .01
.75	4.17	4.15	+ .02
.80	4.12	4.11	+ .02
.85	4.01	4.01	+ .00
.90	3.86	3.84	+ .02
.95	3.68	3.69	- .01

δ Cephei is the characteristic star of the class and has a period of 5.3664 days. The "period" varies slightly, but no account of this has been taken in the plot shown in fig. 3. The variation is only of the order of 10^{-8} days, and hence negligible compared with the inaccuracies in the other data. The radius of the star was calculated from a parallax * of $0''.006$ and an angular diameter † of $0''.0017$, as well as Shapley's period-luminosity law. The effective temperature at maximum light was again taken as an average of five estimates. For comparison, the temperatures at maximum and minimum light are given in Table IV, together with the range, observer and method.

TABLE IV

Observer	Method	Max. Light	Min. Light	Range
Sampson ‡	Spectrophoto	7000°	5800°	1200°
Hopmann §	Colorimetric	6700	4780	1920
Pannekoek and Reesinck	Spectrometric	6400	5400	1000
Petit and Nicholson ¶	Bolometric	6200	4580	1620
Whipple **	Spectrophoto	6650	4500	2150
	Plot	6600	5580	1020

Fig. 3 brings out the fact stated by Whipple that whereas maximum light and maximum temperature approximately coincide, minimum temperature precedes considerably minimum light. The disagreement of the temperatures in Table IV shows the danger of using colour temperatures or even of temperatures derived from spectral type for variable stars.

W Sagittarii exhibits in its radial velocity curve a small but definite secondary maximum. This small maximum occurs at phase 0.5. Its only effect on the locus in the (r, L) diagram (fig. 4), is a slight irregularity, because the maximum in the radial velocity disappears on integration. It disappears as a maximum because the minimum between it and the primary maximum is on the same side of the zero axis of radial velocity. That is, starting at one maximum, the radius is increasing rapidly. At the secondary minimum in the radial velocity curve the radius is still increasing but at a smaller rate. At the secondary maximum the radius is again increasing at a rapid rate. The radius does not start decreasing till the radial velocity curve actually crosses the zero axis. Hence, in general, secondary maxima are not important in determining the number of loops in the characteristic curve unless successive maxima and minima are on opposite sides of the zero axis.

The period of *W Sagittarii* is 7.595 days. Its radius was calculated from Shapley's period-luminosity law. A mean temperature of 5800° was used

* Shapley, *Star Clusters*, 139, 1930.

† Petit and Nicholson, *Ap. J.*, 78, 320, 1933.

‡ Sampson, *M.N.*, 85, 240, 1925.

§ Hopmann, *A.N.*, 226, 1, 1925.

|| Pannekoek and Reesinck, *B.A.N.*, 3, 46, 1925.

¶ Petit and Nicholson, *Con. Mt. Wilson Obs.*, 369, 1928.

** Whipple, *L.O.B.*, 16, 1, 1932.

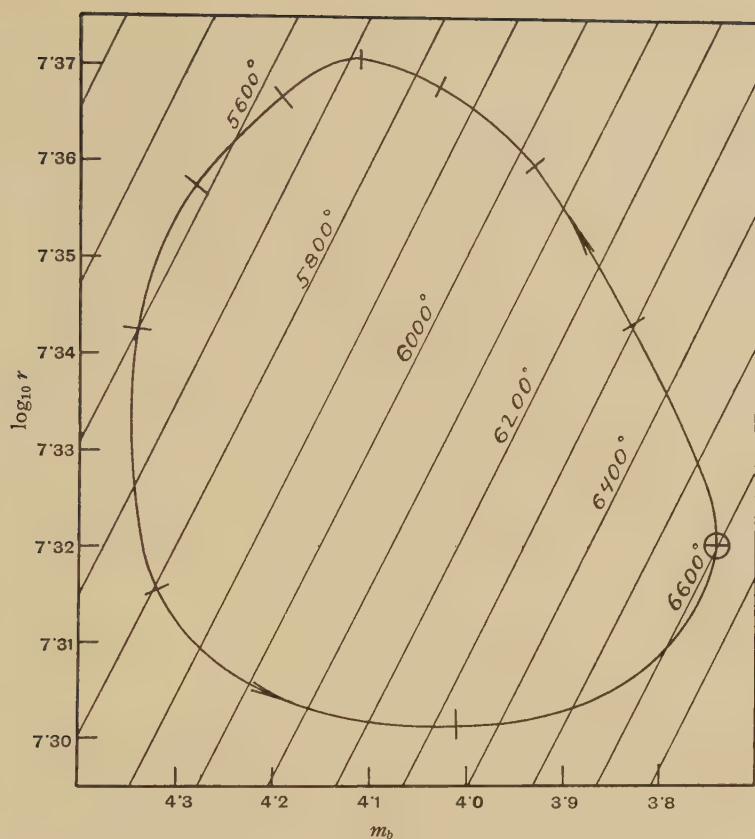


FIG. 3.—Characteristic curve for δ Cephei in terms of effective temperatures and bolometric magnitudes derived from radiometric magnitudes by successive approximations.

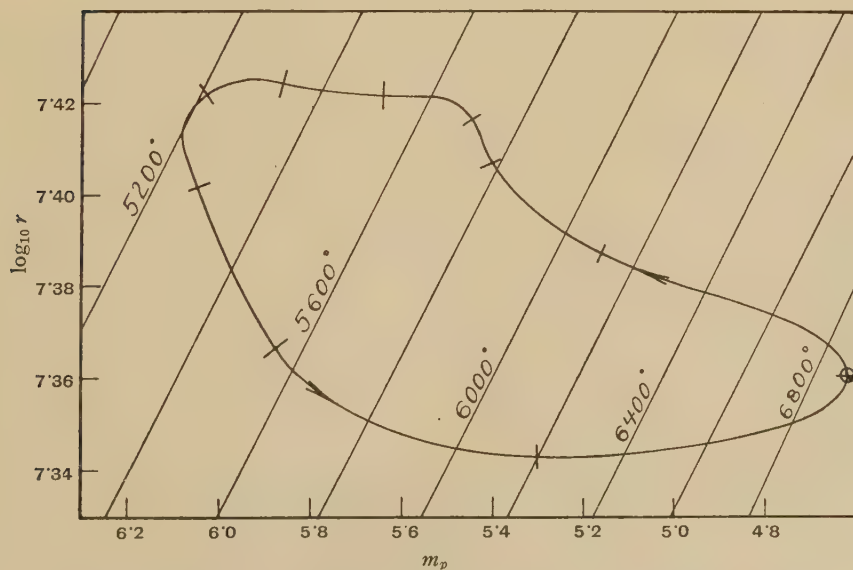


FIG. 4.—Characteristic curve of W Sagittarii in terms of photographic magnitudes and photographic brightness temperature corresponding to the effective wave-length $4250 \text{ \AA}$.

corresponding to the spectral type F8.2.* The spectral variation of F2 to G6 denotes a range of 2500° , but only 1670° was obtained from the plot. In this particular case it is almost impossible to detect any difference of phase between maximum temperature and maximum light, and very little between minimum temperature and minimum light. Substituting appropriate values in equation (5), we get $\delta(\Delta T) = -262^\circ \frac{\delta r_{\min}}{r_{\min}}$. An error of 10 per cent. in the choice of $r_{\min}$ gives therefore a range in temperature variation differing only 26° or about 1.5 per cent.

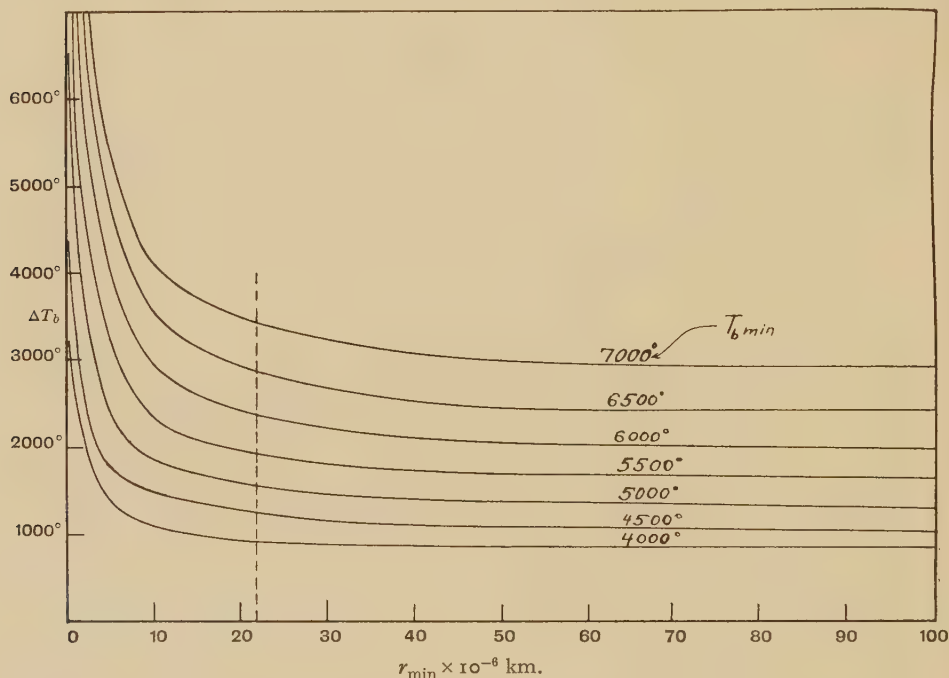


FIG. 5.—The range of temperature variation for *W Sagittarii* as a function of $r_{\min}$ for different values of $T_{b \min}$. Dashed line is the actual value of $r_{\min}$ used.

Fig. 5 shows the range in brightness temperature of *W Sagittarii* for different values of $r_{\min}$ and $t_{\min}$. If spectroscopic type G6 corresponds to 4500° , then from the figure the value of $r_{\min}$ would have to be 3×10^6 km. to give a temperature of 7000° or spectral type F2 at $T_{\max}$. But the radius change from $r_{\min}$ to $r_{\max}$ is 4.64×10^6 km. The change would require a 155 per cent. increase in radius and a ratio of volumes at the extremes of 16.5 to 1. In addition the parallax would have to be very large to give the correct apparent magnitude.

RR Lyræ is a short-period Cluster-type variable of period only 0.5668 days. Its luminosity curve varies, and a mean curve from 1899 to 1930 was used. The radius was estimated from the period-luminosity law. The median

* Shapley and Walton, *H.C.O. Circular*, 313, 1927.

temperature was taken as 8500° , corresponding to the spectral type A5.5.* The variation was found (from fig. 6) to be 1780° as compared to 4700° derived from the spectral type range B9 to F2. Substituting appropriate values in equation (5), we get $\delta(\Delta T) = -310 \frac{\delta r_{\min}}{r_{\min}}$. Hence a 10 per cent. increase in the choice of radius would mean a decrease of about 31° in the temperature range, a change of about 1.7 per cent.

7. *Long-period Variables*.—In contrast to Cepheid variables, all long-period variables for which data was available have characteristic curves traced in the clockwise direction.

o *Ceti* of an approximate period 330 days is the best example of this type. Its radius is given by Pease from interferometer measurements as 2.6×10^8 miles.† The temperature at maximum light is given by Joy ‡ from spectral

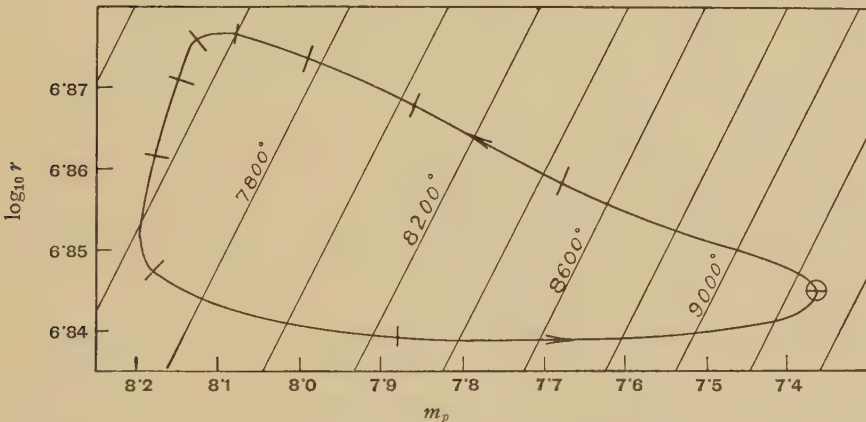


FIG. 6.—Characteristic curve of RR Lyræ in terms of photographic magnitudes and photographic brightness temperatures.

type, by Petit and Nicholson § from radiometric data, and by Merrill || from electric furnace comparisons, as 2400° , 2640° and 2200° respectively. 2640° was arbitrarily selected for the effective temperature at maximum light. Inasmuch as the absorption bands of titanium oxide are not well known in the far infra-red, and as the energy is the greatest in the most doubtful region at the temperature of Me stars, the reduction to no atmosphere cannot be considered very exact. On account of the titanium oxide absorption no correlation can be made between the radiometric and visual plots as was done for η Aquilæ. It has been demonstrated ¶ that if 1800° and 2300° are taken as the temperatures at minimum and maximum light respectively, then for an observer of maximum sensitivity at 5750 A. Planck's law gives a variation of 3.3 magnitudes, and for a radiometer

* Shapley and Walton, *H.C.O. Circular*, 313, 1927.

† Pease, *Proc. A.S. of Pac.*, 37, 89, 1925.

‡ Joy, *Ap. J.*, 63, 281, 1926.

§ Petit and Nicholson, *Ap. J.*, 78, 320, 1933.

|| Merrill, *Ap. J.*, 58, 195, 1923.

¶ Joy, *Ap. J.*, 63, 281, 1926, see p. 330.

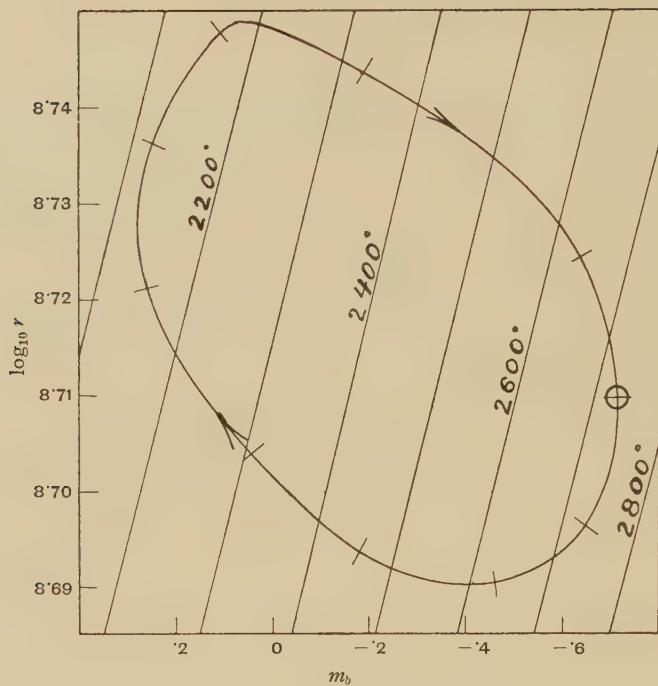


FIG. 7.—Characteristic curve of o Ceti in terms of effective temperatures and bolometric magnitudes derived from successive approximations of the radiometric magnitudes.

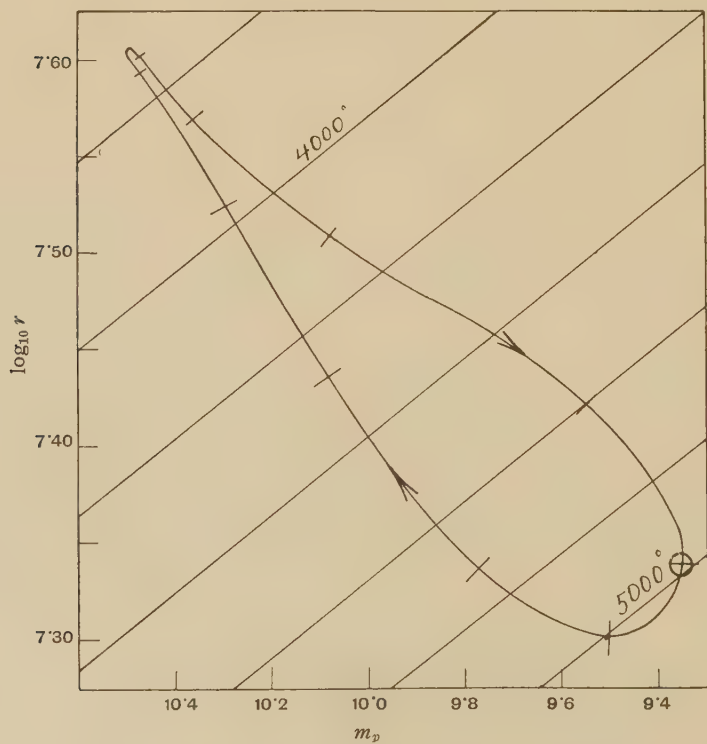


FIG. 8.—Characteristic curve of SX Herculis in terms of photographic magnitudes and corresponding brightness temperatures.

sensitive to all frequencies Stefan's law gives only 1.1 magnitudes. The observed visual magnitude range is 5.3 magnitudes. The difference between the observed and calculated visual ranges is attributed to the titanium oxide absorption bands. The radiometric range is practically the same as the predicted bolometric range. The temperature variation in the plot is found to be 510° as compared with Petit and Nicholson's value of 720° and Joy's 500° .

A question arises in the case of α Ceti in that it has both emission and absorption lines displaced by Doppler effect, the two sets of lines being considerably out of phase with respect to each other, and having even different centre-of-gravity radial velocities. In agreement with standard practice, it has been assumed that the absorption line velocities are representative of the motion of the photosphere.

SX Herculis is among the shortest of the long-period variables, so much so that it has often been compared with Cepheids. Its period is only 103 days. It is very interesting in that it is anomalous in nearly every respect. Like long-period variables in general, it shows titanium oxide absorption bands around minimum light corresponding to spectral types Mo to M₄, whereas its background spectrum is G8 to K₀, or about 1000° higher. It shows emission lines. On the other hand, for a long-period variable its light range is very small, being 1.1 visual magnitudes and 1.2 photographic magnitudes. It is definitely a long-period variable, however, as its characteristic curve is traced in the clockwise direction (fig. 8).

The radius was calculated from its absolute photographic magnitude -1.5 as given by Joy,* and a maximum temperature of 5300° corresponding to its spectral type gG₃. This value of its radius was taken as its minimum, as the maximum temperature nearly coincides with the minimum radius. A variation of 1300° was found as compared with Joy's value 1000° (K₀-G₃). It is interesting to note how narrow the curve is at one end, and to speculate as to the behaviour of stars of somewhat shorter period.

R Virginis, period 145.4 days, has not been sufficiently well worked out to afford a complete plot. However, the incomplete data of Merrill and Joy do show that the direction in the (r , L) diagram is clockwise.

8. *RV Tauri-type Variables*.—AC Herculis and U Monocerotis are both classed together as RV Tauri-type variables in that they both exhibit secondary maxima and minima in their light curves, the secondary properties slowly becoming the primary ones and again returning to secondary status. It is the outcome of this investigation that these stars are not both of the same nature.

AC Herculis has a mean period of 75.4 days with a variation of 6.1 days. Its luminosity curve varies, especially at secondary minimum. An average curve was used in the plot of fig. 9. Phase is measured from the primary minimum of luminosity. The minimum radius was calculated from the full period on the assumption that Shapley's period-luminosity curve could be extended. This does not agree with Gerasimovič's † observation that

* Joy, *Ap. J.*, 75, 127, 1932.

† Gerasimovič, *H.C.O. Circular*, 341, 1929.

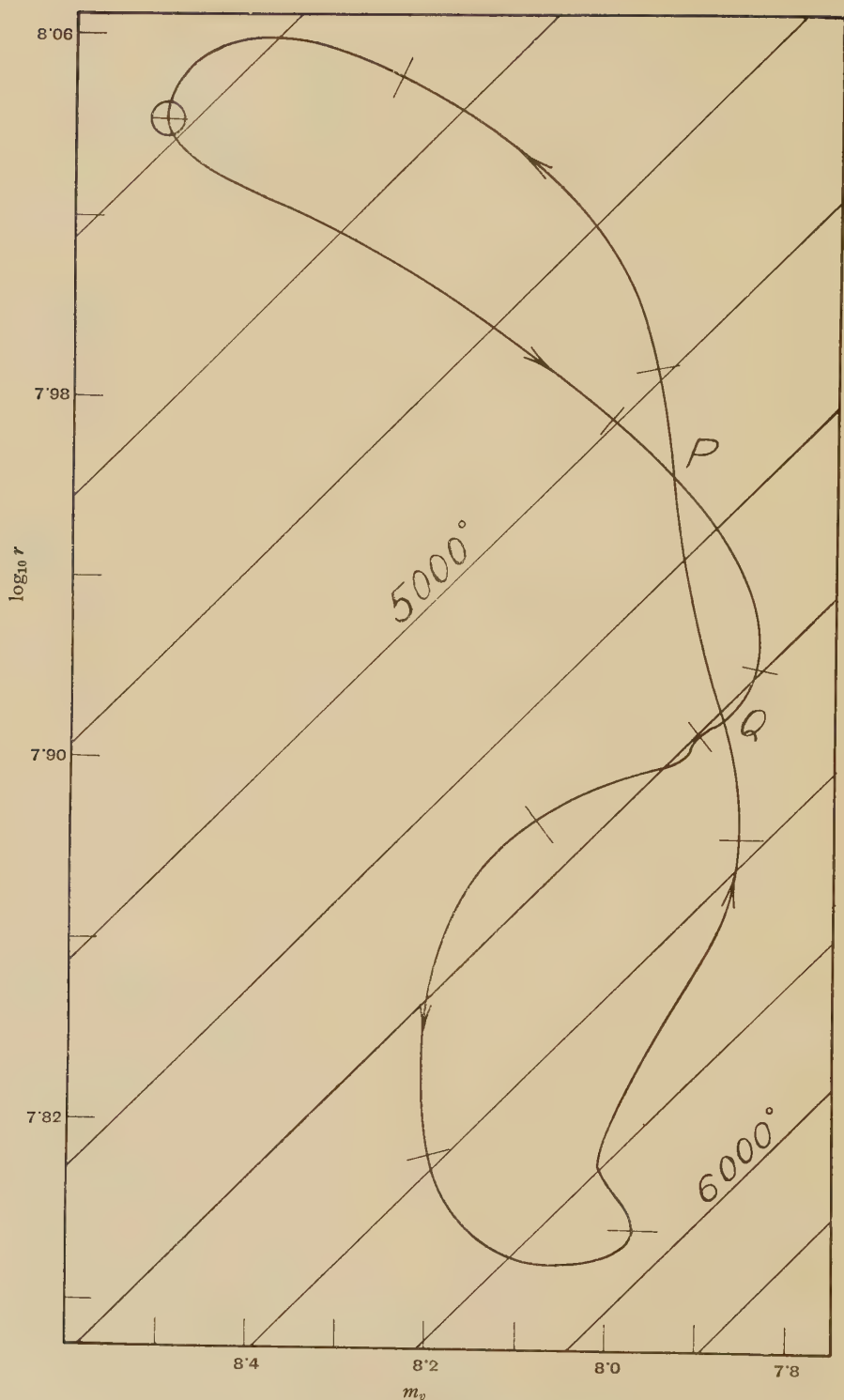


FIG. 9.—Characteristic curve of AC Herculis in terms of visual magnitudes and brightness temperatures corresponding to an effective wave-length of 5290 Å. Phase markings are from principal minimum.

at least in spectral type the RV Tauri stars are related to Cepheids if *half* of the period of the RV Tauri stars is used. However, the spectra of these stars show that the stars are intrinsically very bright giants, which is compatible with the results obtained by the use of the full period. No conclusions that are drawn subsequently depend on the choice of $r_{\min}$. The temperature at $r_{\min}$, taken as 5960° , corresponded to a spectral type gF8 determined by Miss Cannon.* The variation in temperature was found to be from 5980° to 4360° , or 1620° compared to 2500° from Miss Cannon's spectral types, and 3600° from Joy and Sanford's colour indices.† Fig. 10 shows the result

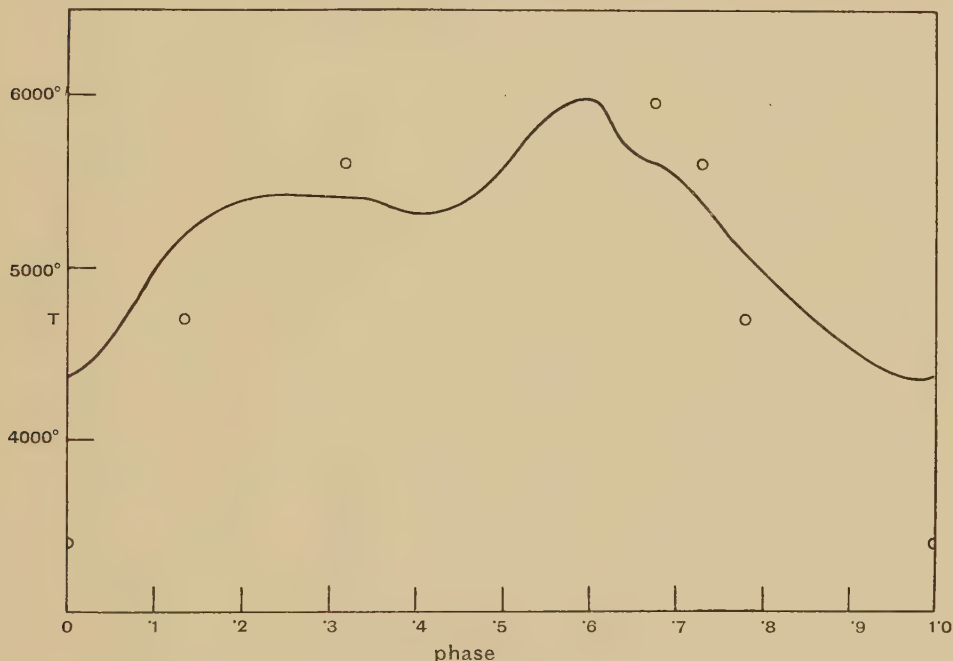


FIG. 10.—Variation in visual brightness temperature of AC Herculis. Circles show the spectroscopic type temperatures.

of reading from the characteristic curve the temperatures as a function of phase. The resulting curve has two maxima, but only one of these is pronounced. The curve agrees in shape with the circles, which show the Harvard Spectral types converted into temperatures. The radial velocities of the arc lines and enhanced lines were different, and had even different centres of gravity. In the plot the arc line velocities were used.

It appears from fig. 9 as if the change from clockwise motion characteristic of long-period variables to anti-clockwise motion characteristic of Cepheid variables occurs not through a general narrowing to an arc, but through an inversion. The interesting points are those where the locus

* Waterfield, *H.C.O. Bulletin*, 845, 11, 1927.

† An arbitrary correction to the visual magnitude curve was made by Joy and Sanford to get approximately correct temperatures. The range is therefore dependent on their correction.

crosses itself. Consider point P, fig. 9. Here the star has a definite value of luminosity L and radius r . Yet at the same configuration it has two different values of radial velocity! At $t=0.12$ the observed radial velocity shows the star to be contracting at the rate of over 13 km. per second, while at $t=0.78$, the star is expanding at nearly 25 km. per second. Likewise at

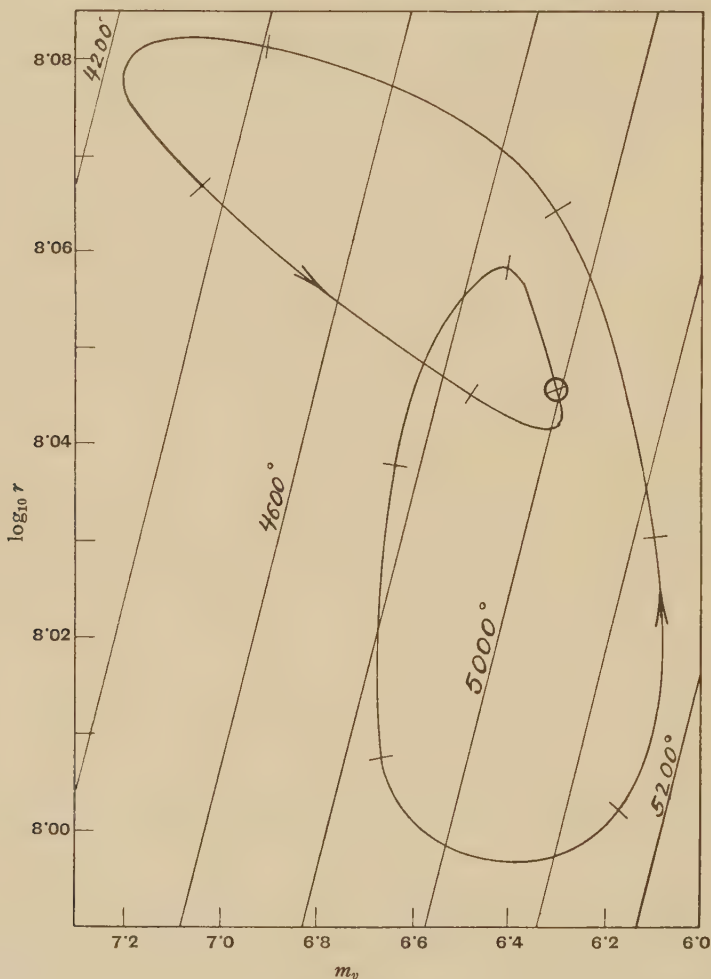


FIG. 11.—Characteristic curve of *U Monocerotis* in terms of visual magnitudes and visual brightness temperatures.

Q , t is about 0.26 and 0.72. At $t=0.26$ the star is contracting with an observed velocity of only 2 km. per second, but at 0.72 it is expanding at the rate of $23\frac{1}{2}$ km. per second.

U Monocerotis has a period of 92.26 days. Its light curve is very irregular. Three periods after the one portrayed in the characteristic curve (fig. 11), the primary minimum and secondary minimum had exchanged places. The radius was calculated from the full period by Shapley's period-luminosity curve. The temperature was fixed at phase 0.3 from principal minimum as

4900° corresponding to type gG4. The radial velocities had been corrected by Sanford for an apparent variation of 40 km. per second of period 2300 days. The comparison of the magnitude of temperature variation with published data is futile, as the light curve varies too much.

The plot of U Monocerotis is entirely different from that of AC Herculis. That of U Monocerotis consists approximately of two deformed ellipses. The direction is always anti-clockwise, as if consisting of two Cepheid loops. The area within the characteristic curve is always of one sign in contrast with the curve of AC Herculis, where the area of the middle section has the opposite sign to the area of the other two sections. Qualitatively one may

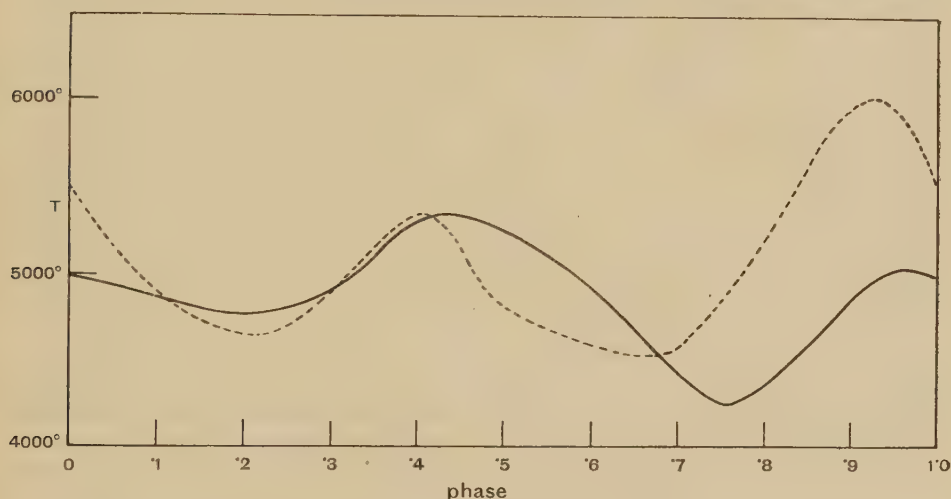


FIG. 12.—Variation of temperature in U Monocerotis. The full line is the visual brightness temperature from fig. 11. The dotted line is the spectroscopic type temperature. Due to the irregular light variations of U Monocerotis, the comparison is only from a qualitative standpoint.

conclude that in the case of AC Herculis the simple harmonics of the radial velocity curve and the luminosity curve are more out of phase than in U Monocerotis.* If temperature is plotted as a function of phase as in fig. 12, there appear for U Monocerotis two distinct maxima and two distinct minima. This again agrees qualitatively with the observed spectral changes.

9. *General Results.*—Milne pointed out in his paper that for regularly pulsating stars the locus of the characteristic point in the (r , L) diagram should be a closed curve, and that the simplest case possible was an ellipse. The present work shows that in many cases the curve actually consists of a single loop. This circumstance leads to a simple method of classification of all regular variables into three classes: (1) stars which have one simple loop in their characteristic curve traced in the anti-clockwise direction; (2) stars which have one simple loop in their characteristic curve traced in the clockwise direction; and (3) stars possessing complicated characteristic curves. All Cepheid variables, including the Cluster-type variables, are of

* E. A. Milne, *M.N.*, 94, 418, 1934.

the first type. All long-period variables seem to be of the second type. RV Tauri-type stars come under class three. There has not been found any star for which the moving point retraces an arc in the (r, L) diagram, and therefore no star satisfying the simplified pulsation theory of Eddington. Milne predicted that the transition from the anti-clockwise to the clockwise plots should occur through a collapse of the oval into a single arc traced and retraced. From this one might have expected that near the border line of Cepheids and long-period variables, that is, for stars of period of about 60 days, the curves should have negligible area. The unexpected result of actual plots is that complications arise. It appears as if some mysterious hand grasped one side of the curve and tried to pull it through the other side as if turning a glove inside out! However, even this is not a general case for stars of the transitional type, for U Monocerotis exhibits a multiplicity of loops rather than the inversion effect just noted. The lack of sufficient data on RV Tauri-type stars prohibits the further study of this interesting class.

The time spacings along the paths show that the velocity of the representative point around the characteristic curve is never very constant. In Cepheids, half of the cycle is completed in 0.7 of the period. The characteristic curves are never ellipses, nor can they be expected to be so, as the light curves are never symmetrical.*

It has been tacitly assumed in this paper that radial velocities as determined from absorption line displacements are representative of the motion of the photosphere. Baade † suggested that the existing data on radial velocities, luminosity and colour indices were sufficient to test the pulsation theory. Working along these grounds, Whipple ‡ has shown that the pulsation hypothesis is insufficient to account for the large changes of temperature, whereupon he invoked Menzel's § theory of "hot spots." Bleksley || has constructed an (r, L) diagram of RS Boötis, calculating the relative radius as a function of phase from colour index temperatures and light curves on the basis that these data are more representative of the photosphere than the spectroscopic radial velocities. He finds for RS Boötis a figure-eight curve not at all consistent with the statement in this paper that all Cepheids have characteristic curves of one loop traced in the anti-clockwise direction. While there is considerable doubt as to the exact interpretation of the Doppler shift as radial velocities of the photosphere, ¶ the use of colour indices in terms of temperature, particularly for variable stars, is in the opinion of the author more hazardous. Since colour temperatures in general can vary above or below effective temperatures or brightness temperatures, it is much

* Milne has shown on certain assumptions that the light curves cannot be symmetrical (*Quarterly Journal of Mathematics*, Oxford Series, 4, 258, 1933).

† Baade, *Hamburg Mitt.*, 6, 85, 1926.

‡ Whipple, *L.O.B.*, 16, 1, 1932.

§ Menzel, *Popular Astronomy*, 39, 16, 1931.

|| Bleksley, *Nature*, 134, 661, 1934; *Observatory*, 57, 343, 1934.

¶ Petit and Nicholson (*Ap. J.*, 78, 323, 1933) give curves of spectroscopic and radiometric radial velocities of α Ceti nearly completely out of phase. Furthermore, emission-line velocities are nearly always out of phase with absorption line-velocities.

more dangerous to compare them in a variable star whose outer layers are changing in unknown ways and whose spectra themselves are anomalous.*

Hence, if the radial velocities do measure the motion of the photosphere, and relying on the fact that corrections to bolometric magnitude are not sensitive to distortions of the frequency distribution curve from that of a black body, one may say that the variation of effective temperature as found from the bolometric (r , L) diagram is representative of the magnitude of the variation, and that a complete theory of the pulsating-radiative equilibrium of the photosphere of a variable star must explain why both the colour temperature and spectroscopic types give larger variations. Ionization temperatures are of course rather difficult to apply to variable stars whose density is changing.

10. *Summary.*—Milne's proposed method for the analysis of variable stars of variable radius has been carried out for actual examples. A classification of variables is suggested according to the direction in which the representative point of the star moves in the (r , L) diagram. The (r , L) diagram of four Cepheids, η Aquilae, δ Cephei, W Sagittarii and RR Lyræ, are given together with the variation in temperature. The characteristic plots of the long-period variables α Ceti, SX Herculis and R Virginis have their curves traced in the clockwise direction as contrasted with the Cepheids whose points move in the anti-clockwise direction. The RV Tauri variables AC Herculis and U Monocerotis, stars of intermediate period, are shown to have distinctly different phase relations. It is shown that a star of fixed mass, and possessing the same value of radius and luminosity at two different phases, may still have different values of surface velocity, pointing to the implication that other physical quantities, like the internal entrapped energy, may not be single-valued functions of the radius. It is pointed out that the pulsation hypothesis predicts a temperature curve for the RV Tauri-type stars qualitatively similar to actual spectroscopic type temperatures. The general disagreement between colour temperatures and spectroscopic type temperatures as compared with those predicted by the (r , L) diagram is attributed to the lack of sufficient understanding of the physical processes at the surface of a pulsating star rather than to the failure of the pulsation hypothesis itself.

The author is deeply indebted to Professor E. A. Milne and to Professor H. H. Plaskett, of the University Observatory, Oxford, for their generous assistance and guidance.

University Observatory, Oxford :
1934 November.

* Among these may be noted the abnormality of the hydrogen lines and the fact that the spectral type varies for Cepheids on the average from F1 to F7 for hydrogen lines, while the general background varies only from F9 to G0 (Adams and Joy, *Proc. Nat. Acad. Sci.*, 4, 129, 1918).

A MOLL-TYPE MICROPHOTOMETER AND ITS PERFORMANCE.

H. H. Plaskett.

The recording microphotometer designed by Moll * differed from the original Koch instrument † in two essential features. In the first place a thermocouple with galvanometer was used as the photometric receiver in place of Koch's photo-electric cell and electrometer, thus securing a constant and linear relation between incident light and recorded deflection; and in the second place a slit reduced the illuminated area on the photographic plate to the exact dimensions of the analysing beam, thus eliminating the error due to scattered light discovered by Schwarzschild and Villiger. ‡ Actual experience with the Moll microphotometer at Harvard § and at Lick, while fully confirming the advantages claimed for this type of instrument, has revealed certain imperfections in mechanical design and in exactness of adjustment. Advantage therefore has been taken of the opportunity of acquiring a microphotometer for the University Observatory, Oxford, to design an instrument of this type in which some of these difficulties have been avoided. This instrument, jointly designed by R. M. Abraham and the writer, and constructed by Casella & Co., has proved to be not only relatively inexpensive, but also accurate in use and easy of adjustment.

The present paper contains in the first section a brief description of the instrument and its adjustment, and in the second section an account of tests of the instrument, especially with regard to proportionality of linear displacement and resolving power. It is hoped that these tests may prove to be sufficiently revealing, though simple of execution, to justify their application to other microphotometers.

Section I.—The Instrument and its Adjustment

I.1. The Instrument.—The optical system of the microphotometer is essentially identical with Moll's original design. The lamp house with its Zeiss condensing lens ($\times 8$) forms one unit, A (see Plate 2, fig. 1); a second unit, B, consists of one of four self-centring interchangeable slits ruled in silver on glass and a Zeiss apochromatic objective (primary magnification 10, numerical aperture 0.30), both mounted in the one microscope with tube length variable between 13 and 15 cm.; and the third unit, C, has an identical microscope objective and the Moll quick vacuum

* *Proc. Physical Soc. London*, 33, 207, 1921.

† *Ann. d. Phys.*, 39, 705, 1912.

‡ *Astroph. Journ.*, 23, 286, 1906.

§ H. H. Plaskett, *M.N.*, 91, 882, 1931.

|| A. H. Farnsworth, *L.O.B.*, 16, No. 456, 1933.

thermocouple (period 0.3 second) with horizontal and vertical adjustments within the mount but without the usual subsidiary slit, field lens and objective. Provision is made in each of these units for focussing, for independent rotation of the parts, and for placing the lamp filament, slit or thermocouple on the optical axis of the condensing lens or microscope objectives respectively. Each unit is held to a geometric slide, which is free to move or can be clamped at any desired position, on the optical bench, I, by two sets of three collimating screws which press on two coaxial circular rings on the unit.

The photographic plate, which can be taken in sizes up to 12.5×25 cm. (5×10 in.) without cutting, is held by cork-tipped fingers to an inner stage which is adjustable for height. This inner stage is supported by flat leaf-springs under tension on the main stage, E, which in turn is pivoted to the carriage; this mode of support ensures that a straight line on the plate can readily be made parallel to the direction of motion of the carriage. The carriage is another geometric slide which moves on ways consisting of two lapped silver steel rods with centres 16 cm. (6.5 in.) apart; this broad base ensures that any microscopic irregularity in the ways produces the minimum disturbance in the motion of the plate. Four springs and rollers support three-quarters of the weight of carriage and stage, thus reducing wear at the five constraints (two *V*'s and a plane) and ensuring a smooth easy motion for the carriage on its ways. If desired the carriage and stage can be lifted as a unit off the ways.

The carriage is driven in one direction (away from the gear system) by the pressure of a bronze nut with a hardened steel abutment. This nut is carried forward by the rotation of its screw of tool steel, one-inch in diameter, with ten threads to the inch, and cut over a length of six inches; the screw is held between 60° centres, as in cutting and grinding, and has on its shaft a small gear which engages with the gear F. The gear F may be turned either directly by the handle seen on the same spindle, or, by engaging the handle by means of a screw to the worm wheel, the gear F is turned by a motor drive through the pulley. The nut is prevented from turning with the screw, and so is moved forward to push the carriage, by a hardened steel finger from it which bears on a straight edge; this latter could, if necessary or desired, serve as a corrector bar. The motor gives plate speeds ranging from 3 cm. to 24 cm. per hour; the latter of these is nearly the maximum speed which should be used if thermocouple and galvanometer inertia is not to distort the record.

The gear F also serves, through the idler gear G, to drive the spindle which carries the gear H, which spindle is in line with and connected by means of a key to the hollow drum shaft. By means of a motion of this spindle either H may engage with a small gear which can be seen in the plate on G's spindle, in which case one revolution of the drum occurs for twenty-eight revolutions of the screw, or a gear behind H may engage directly with G as in the plate, in which case one drum revolution occurs in four screw revolutions. The actual magnification of the record on the drum over that of the object on the plate depends, of course, on the radius of

the drum; the magnifications corresponding to these two spindle positions are, closely enough, 6.89 and 48.2.

The drum is provided with the usual slit and cylindrical lens, fixed in the lower case which carries the drum bearings. The excursion of the slit image given by a Moll micro-galvanometer (period 0.25 second), which is connected directly with the thermocouple, is recorded as a black line on a sheet of bromide paper 12.5×50 cm. (5×20 in.), clamped on the drum. In order to set up a co-ordinate system on this bromide paper record, which is independent of any subsequent changes in the dimensions of the paper on developing and fixing, and which is independent of errors in the gear train, a subsidiary lamp is erected beside the galvanometer and intermittently lit by a 6-volt transformer through the contact maker, to be seen in Plate 2 to the left and behind the gear F. This contact maker is actuated by one of two special gears on the screw shaft which are cut with ratchet teeth; by moving the contact maker to engage with the appropriate ratchet tooth gear black lines are printed on the record parallel to the drum axis for definite fractions of a screw revolution, and at the same distance apart, roughly 8.7 mm., for either magnification. A series of lines also at the same distance apart, but at right angles to the time marks, is produced on the record by means of fine slits engraved on a silver on glass scale set in the drum cover and continuously illuminated by a small lamp vertically above the scale. Fig. 2 (Plate 2) shows part of a microphotometer record of the solar spectrum near $H\gamma$ with this superposed grid [dispersion of spectrum 16 Å. per mm.; microphotometer slit 10.4μ ; magnification of record 48.2].

1.2. *Collimation*.—On first setting up the microphotometer, and subsequently only if the lamp or the objectives are changed, it is necessary to collimate the optical system. Accurate collimation ensures first the critical definition, only attainable on the optical axis, which is necessary to reach the optimum resolving power; secondly, the maximum galvanometer deflection for a given energy output of the analysing lamp; and finally a symmetrical distribution of stray illumination, due to reflection off the various lens surfaces, about the analysing beam. A characteristic of the present instrument is the accuracy and permanence of its collimation.

In adjusting the instrument for collimation the following procedure has been found convenient:—

Unit A.—Rotation of the lamp house as a unit about the optical axis of the condensing lens immediately discloses, by a displacement of the image, any departure of the centre of the filament from this axis. Such departure may be corrected by the collimating screws which hold the lamp in position. When the lamp house has been collimated as a unit it only remains to make its optical axis parallel to the optical bench. A departure from parallelism is revealed by a displacement of the filament image as the lamp house on its slide is moved along the optical bench, and may be corrected by the six collimating screws which hold the lamp house to the slide.

Unit B.—Since the definition of the slit image, and consequently the resolving power of the microphotometer, depend upon accurately centring

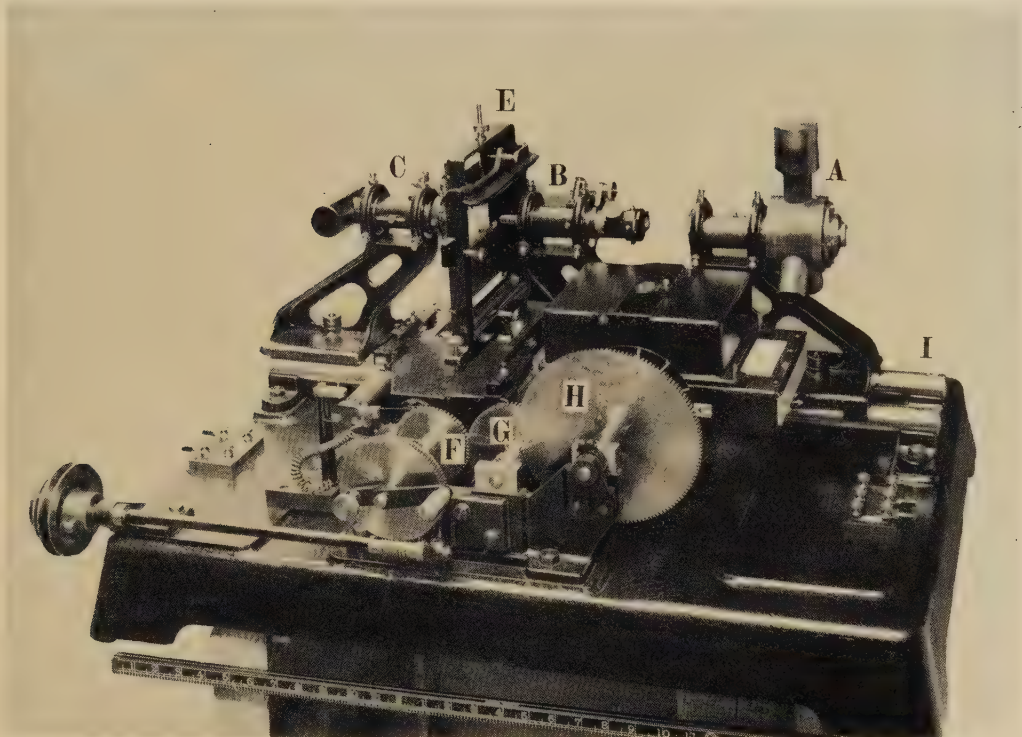


Fig. 1

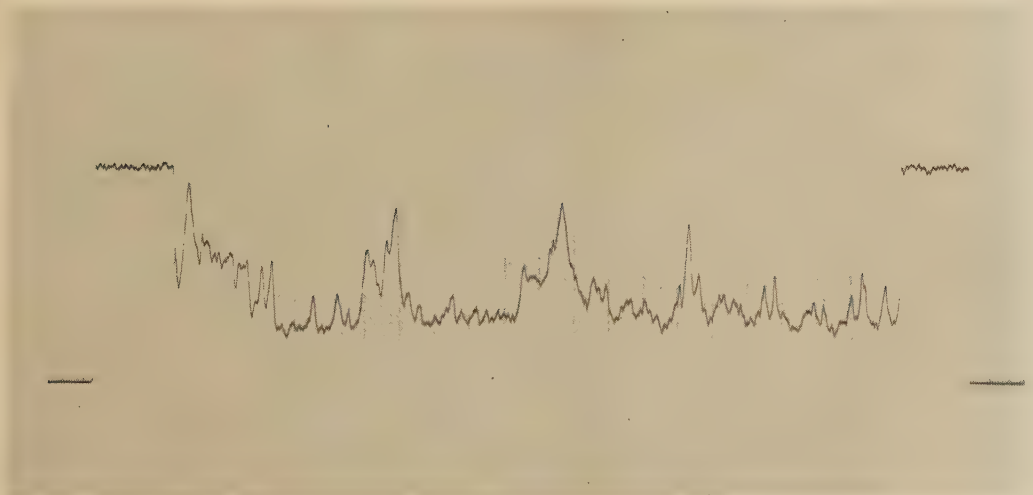


Fig. 2

H. H. Plaskett, A Moll-type Microphotometer.

the slit on the optical axis of the objective, a more sensitive test than is afforded by rotation is required for detecting lack of collimation. The first step in the adjustment is by varying the tube length to make the linear separation of the images of two marks on one of the slits exactly one-tenth their actual separation; since the primary magnification of the objective is ten, this procedure ensures that the slit is at the optical tube length for which the objective is corrected, and that, consequently, when the slit is placed on the optical axis its image will be critically defined. The second step is to examine this image, as formed on a plate, with a low-power magnifier. If the slit is not centred, then surrounding the bright central image, but lying in slightly different planes, will be found a number of fainter images which arise from internal reflections within the microscope objective. Accurate centring is then secured by adjusting the collimating screws which hold the slit until the principal and the internally reflected images coalesce. The final step is to make the optical axis of the now collimated slit microscope coaxial with that of the lamp house; this may be simply effected by adjusting the microscope as a unit until the filament image is centred on the slit, and at the same time, when the lamp house has been moved back on the optical bench, the narrow band of the extra focal patch, as limited by the slit, is centred on the front face of the microscope objective.

Unit C.—Since in this type of microphotometer definition and resolving power are essentially determined by the collimation of units A and B, it suffices to adjust unit C until the galvanometer deflection is a maximum, and then by means of the collimating screws which hold the unit to its slide to make its axis coaxial with that of the common axis of units A and B.

1.3. Plate Adjustment.—Before a photographic plate, carrying for example a spectrum, can have its transmission recorded by the microphotometer, it is necessary to be able accurately and quickly to align the spectrum and sharply to focus the slit image on the plate. With the aid of a positive eyepiece behind the thermocouple, with which can be seen the linear thermocouple strip and any real images formed in its plane by the objective of unit C, this alignment proceeds as follows:—

First microscope B is put out of focus to give a broad illuminated patch on the plate, and microscope C is focussed until the thus illuminated silver grains are seen through the eyepiece to be sharply focussed in the plane of the thermocouple. Then the plate carriage is moved in its ways, and, if necessary, the screws working against the flat leaf-springs, which hold the inner stage and the screw which determines the tilt of the main stage, are adjusted until the silver grains remain sharply in focus and the spectrum remains centred on the thermocouple strip through the whole motion of the carriage. Thus the spectrum is made strictly parallel to the direction of the carriage motion. Next, by means of a tangent screw, the thermocouple mount is tilted in unit C until the linear junction is parallel to the spectrum lines, and then the microscope B is focussed until the slit image is sharply defined, when, if necessary, the slit can be rotated in unit B until its image is parallel to the thermocouple. Finally, by means of the horizontal and vertical adjustments in the thermocouple mount, the linear junction is moved parallel

to itself until the slit image is centred on it. The whole operation can be carried out both quickly and positively, and so satisfactorily do the apochromatic objectives define in both visual and infra-red that no change of these adjustments increases the galvanometer deflection.

Section 2.—Performance

A recording microphotometer for which the claim of precision is made should have three characteristics. It should give a relation between plate transmission and deflection which is independent of the time, or at least strictly reproducible; it should give an exact proportionality between displacement measured on the bromide paper record and displacement on the plate; and finally it should have high resolving power. Tests of the present instrument for these characteristics will now be described.

2.1. *Characteristic Curve.*—Since the photographic plate scatters rather than absorbs the light incident on it, the flux, T , transmitted by a given area depends upon the angular aperture and effective wave-length of the analysing beam. If, and only if, these variables are held constant, is T an invariant physical quality, and is equality of silver deposit measurable with the microphotometer. Assuming this to be the case, and it is easily effected, let the deflection which corresponds to T be y , measured on the record from the zero line ($T=0$), and let y_0 be the similarly measured deflection which corresponds to the unexposed part of the photographic plate. Then the functional relation between y/y_0 and T may be called the characteristic curve, and dy/dT the sensitivity of the instrument. For a given microphotometer in which the angular aperture and effective wave-length of the analysing beam are held constant, the characteristic curve and the sensitivity depend wholly on the characteristics of the photometric receiver and its associated electrical circuit.

In the present instrument the photometric receiver is a thermocouple with galvanometer, giving a linear characteristic curve. Such a receiver has the great advantage that its characteristic curve is independent of time, so that it is not necessary, as Lindblad and Stenquist * found it necessary with the Stockholm Koch-Goos microphotometer, repeatedly to standardize the instrument through the use of filters of differing transmission. The characteristic curve of this thermocouple, as contrasted with those of two photo-electric receivers, is shown in fig. 3, where the abscissæ are values of y/y_0 for the present instrument, and the ordinates of the filled circles are the corresponding y/y_0 for a Zeiss microphotometer, while the ordinates of the half-filled circles are the same quantities for a Cambridge microphotometer. The data for this figure were obtained from records made at Dr. Redman's suggestion and recently discussed by him.† The comparison, as a comparison of characteristic curves of different receivers, is only approximate because a photographic plate was used as a transmission standard, and because the angular aperture and effective wave-length of the analysing

* *Astronomiska Iakttagelser å Stockholms Observatorium*, II, No. 12, 1934.

† *The Observatory*, 57, 275, 1934.

beam were different in the three instruments. The figure gives, however, at least a qualitative idea of the performance of the receivers, and in particular shows the intentionally arranged high sensitivity of the Zeiss instrument for low transmissions as contrasted with the high sensitivity of the Cambridge instrument for large transmissions. From several points of view the linear characteristic curve of the thermocouple has something to commend it,

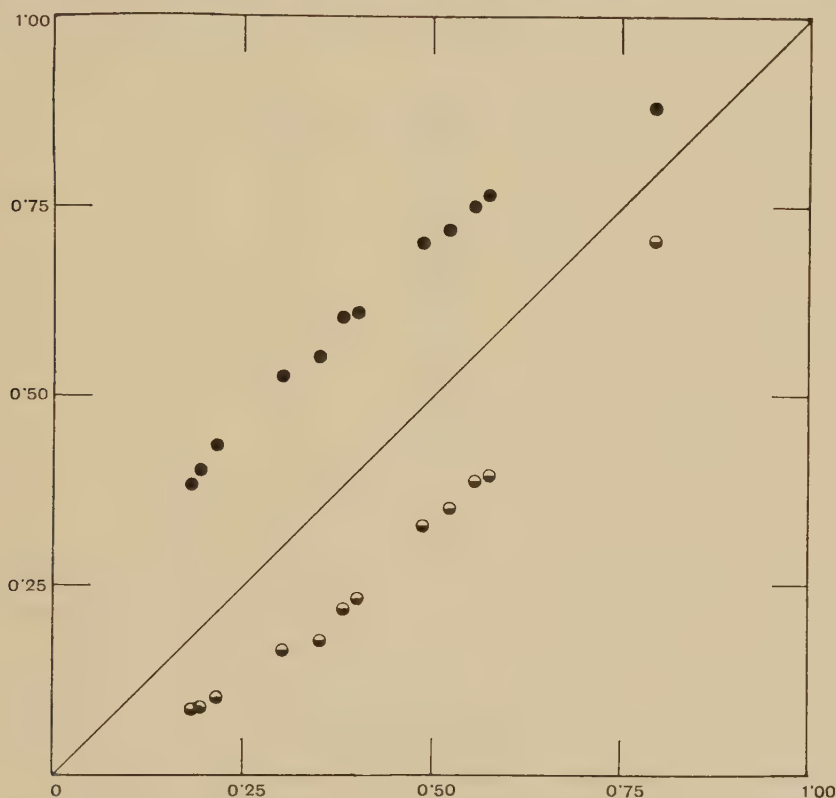


FIG. 3.—*Relative Characteristic Curves.* Abscissæ are values of y/y_0 for Oxford Microphotometer, and ordinates of the filled and half-filled circles are the corresponding values of y/y_0 for the Zeiss and Cambridge Microphotometers respectively.

but the chief advantage of the thermocouple undoubtedly lies in the constancy of this characteristic curve with time.

2.2. *Proportionality of Displacement.*—To obtain proportionality between displacements on the plate and the record to the requisite degree of accuracy imposes the severest demands both on the gear system and on the screw and straightness of the carriage ways. In order to test the gear system, records were made of the time marks, which, as has been described, are printed on the record for equal fractions of a revolution of the screw. If the bromide paper has not changed dimensions on development, and if the gear system is accurate, these time marks should be equally spaced. A sample of the results from the measures of one record is given in Table I, under the heading of Model I. The first column gives the number, n , of

TABLE I
Gear and Drum Errors

<i>n</i>	Model I				II	IA
	x	Obs.	Comp.	O - C	O - C	O - C
	mm.	mm.	mm.	mm.	mm.	mm.
0	13.8	13.8	13.8	± 0.0	± 0.0	± 0.0
4	48.7	48.8	48.8	± 0.0	-0.7	± 0.0
8	83.7	83.8	83.8	± 0.0	-1.2	+0.1
12	118.9	119.0	118.8	+0.2	-2.1	+0.2
16	154.3	154.5	153.8	+0.7	-2.3	+0.2
20	189.6	189.8	188.8	+1.0	-2.4	+0.2
24	225.2	225.4	223.8	+1.6	-2.3	+0.4
28	260.6	260.8	258.8	+2.0	-1.8	+0.5
32	295.8	296.1	293.9	+2.2	-1.2	+0.5
36	330.3	330.6	328.9	+1.7	-0.4	+0.3
40	364.7	365.1	363.9	+1.2	± 0.0	+0.1
44	399.2	399.6	398.9	+0.7	+0.2	+0.3
48	433.7	434.2	433.9	+0.3	+0.4	+0.3
52	468.3	468.9	468.9	± 0.0	± 0.0	± 0.0

the time mark, the second its measured x in mm., the third this measured x corrected for a change in dimensions of the bromide paper (correction found from a pencil grid ruled on the back of the record and measured before and after development), the fourth column gives the position of the time mark computed from the expression $n(x_{52} - x_0)/52$, and the fifth column gives the residual O - C. From the run of the residuals, confirmed from other records, it was clear that there was eccentricity either in the drum or gears of Model I. Consequently Model II with a newly designed drum, and with gears mounted on heavy steel hubs on their several pinions and with teeth cut on a dividing engine with an accuracy of three seconds of arc, was put into operation in March. O - C residuals, derived in the same way as for Model I, are given under the heading II in the sixth column of Table I; it will be observed that, in spite of these precautions in construction, the errors in Model II are as large as those in Model I. From a scale engraved on the large gear, H, it was possible to show that of these large residuals only about 0.3 mm. was contributed by gear errors, and since the drum was accurately centred and turned, it followed that the residuals in the case of Model II, and possibly Model I, must have originated in the coupling between the gear spindle and the drum shaft.

This coupling is shown in section in fig. 4. Here O is the centre of the hollow drum shaft, O' is the centre of the gear H spindle, and A is a key attached to the spindle which, when the spindle is moved perpendicular to the plane of the diagram in order to change gears, moves in a slot or key-way in the hollow shaft. If the spindle and shaft are not exactly lined up,

that is $OO' = b \neq 0$, then when the spindle has turned through $\pi/2$ so that the key is at B, the drum shaft will have only turned through the angle $\frac{\pi}{2} - \sin^{-1} \frac{b}{r}$ (or, approximately, $\frac{\pi}{2} - \frac{b}{r}$), where r is the radius of the hollow drum shaft. Hence the drum will be out of position in angle by b/r , and the record will be linearly out of position by Rb/r , where R is the radius of the drum. From the dimensions of the parts it follows that the residuals shown for Models I and II would be produced if $b = 0.13$ mm. (0.005 in.), a rather small quantity.

When it was realized where the source of the trouble lay, Model I was rebuilt with new gears, cut as in Model II, only with finer teeth so that gear

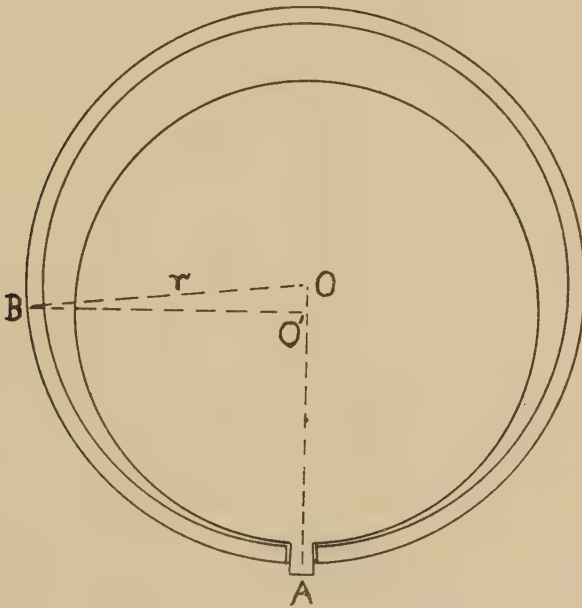


FIG. 4.—Effect of Eccentricity of Gear Spindle in Drum Shaft.

errors contributed less than 0.08 mm. to the residuals, with a new drum and with the drum shaft and gear spindle carefully aligned with the aid of a true arbor. In the last column of Table I the residuals obtained with this instrument, described as Model IA, are given from measures of the same type as for the two previous models. From these residuals—and five other records showed residuals smaller than these—it will be seen that the error between the screw and the drum has now been reduced to a satisfactorily small quantity—a quantity which is indeed comparable with the errors introduced by shrinkage and expansion of the bromide paper.

2.21. Having assured ourselves that the gears and drum in Model IA are sufficiently accurate, it now remains only to test the accuracy of the screw and the straightness of the ways. For this purpose a glass scale, ruled with fine divisions every tenth mm., was standardized at a number of divisions from measures with a Hilger micrometer by the method of repeated

coincidences.* Table II contains in its first column the division number (in mm. from an arbitrary origin in the scale), and in the second column the correction in microns to be added to this division number to give the actual position of the division from the arbitrary origin; the corrections are the means from three independent sets of measures, and the attached errors are the standard deviations of these means. This standardized glass scale was set up on the stage of the microphotometer, and three records with a

TABLE II
Standard Scale

Divn.	Correction	Divn.	Correction
mm.	μ	mm.	μ
0	$\pm 0.0 \quad \pm 0.0$	25	$\pm 0.0 \quad \pm 0.0$
9	$-1.4 \quad .3$	26	$-0.7 \quad .0$
18	$\pm 0.0 \quad .5$	27	$-0.7 \quad .3$
27	$+0.1 \quad .5$	28	$+1.0 \quad .2$
36	$+1.6 \quad .3$	29	$-0.1 \quad .2$
45	$+0.5 \quad .3$	30	$+1.7 \quad .3$
54	$\pm 0.0 \quad \pm 0.0$	31	$\pm 0.0 \quad \pm 0.0$

magnification of 6.89 obtained. These records were measured from the co-ordinate system imprinted by the time marks, thus automatically eliminating any residual errors due to gears, drum, alignment and bromide paper; a glass reticule, accurately ruled for the specific purpose of measuring records made with Model IA, made measurement precise and rapid. The results are given in Table III, where the first column contains the position of the scale divisions on the glass scale (taken from Table II), the next three columns contain the measured positions of these divisions on three records (1, 2, 3) in units of time marks, and the fifth column contains the mean from these

TABLE III
Screw Errors with Magnification 6.89

Scale	1	2	3	Mean	Comp.	O - C
mm.					mm.	mm.
0.000	0.00	0.00	0.00	0.00	0.000	$\pm .000$
8.999	7.08	7.09	7.09	7.09	9.002	$-.003$
18.000	14.18	14.18	14.17	14.18	18.004	$-.004$
27.000	21.27	21.27	21.26	21.27	27.007	$-.007$
36.002	28.35	28.36	28.35	28.35	35.996	$+.006$
45.000	35.44	35.43	35.44	35.44	44.998	$+.002$
54.000	42.53	42.53	42.54	42.53	54.000	$\pm .000$

* D. Gill, *M.N.*, 49, 105, 1889; H. Jacoby, *Am. Journ. Science*, 151, 333, 1896.

three records. Since 42.53 time marks on the record is equivalent to 54.000 mm. on the standardized scale, 1 time mark is equivalent to 1.2697 mm., and, using this factor, the mean measured positions on the records are converted into computed positions on the glass scale. The last column of the table contains the difference (O - C) between the actual positions, as given in the first column, and these computed positions. Since the error in measurement is 0.01 of a time mark, residuals up to 0.013 mm. may be expected from this source alone; it will be noted that no residuals exceeding this quantity are found.

A more critical test is afforded by using the 48.2 magnification where the proportionality factor is 1 time mark = 0.1814 mm., and where consequently the accidental error of measuring a record, projected on the glass scale, amounts only to 0.0018 mm. Three records were made of the central part of the glass scale (divisions 25 to 31), one with the nut at the beginning of the screw, one with the nut near the middle, and one near the end of the screw. The results are given in Table IV where the first column gives the division position (taken from Table II), and the next three columns are the O - C residuals in microns (found as in Table III) from these three records (numbered 4, 5, 6). It will be noted that with the exception of division 27, and possibly division 30, the residuals are less than the accidental error of measurement. Since for the two exceptions the residual has the same sign

TABLE IV

Screw Errors with Magnification 48.2

Scale	Residuals O - C			
	4	5	6	Mean
mm.	mm.	mm.	mm.	mm.
25.0000	± .0000	± .0000	± .0000	± .0000
25.9993	+ .0004	- .0017	- .0015	- .0009
26.9993	- .0028	- .0027	- .0038	- .0031
28.0010	- .0011	+ .0004	+ .0002	- .0002
28.9999	+ .0001	+ .0001	- .0004	- .0001
30.0017	+ .0012	+ .0022	+ .0019	+ .0018
31.0000	± .0000	± .0000	± .0000	± .0000

and roughly the same size at three different parts of the screw and ways, it is not improbable that some of the error is inherent in the scale rather than in the instrument. Thus a slight irregularity in the carbon filling of a division mark, within 0.2 mm. of the height on the scale at which the records were made, would be recorded by the microphotometer, but would be averaged out in the visual standardization with the Hilger micrometer. Whether this be the interpretation or not, it is clear that the screw and the ways of the microphotometer have an accuracy of at least $\pm 3 \mu$, possibly $\pm 2 \mu$.

Since the errors in the screw and the ways of the instrument are of this order it follows that, from displacements measured with reference to the time marks over, if desired, the full 50 cm. of the record, the displacement computed on the plate will also be accurate to at least $\pm 3 \mu$, possibly $\pm 2 \mu$. This accuracy, it may be pointed out, is not far removed from the maximum accuracy which can be reached from direct measurement of a photographic plate, since distortions in the emulsion produce spurious displacements of the order of ± 1.6 .*

It is of interest to compare the accuracy of this microphotometer with that of others on which actual tests have been carried out. The results of these tests are summarized in Table V. In each case the test has been made against a measured scale, but only in the present instance has this scale been

TABLE V

Errors of Microphotometers

Instrument	Error	Recording Drive	Observer	Reference
	μ			
Goos	± 0.1	Rotating mirror	F. G.	<i>Zs. f. Inst.</i> , 41, 313.
Koch-Goos	1	Steel bands	F. G., P. P. K.	<i>Zs. f. Ph.</i> , 44, 855.
Oxford	2.5	Gears	H. H. P.	Present paper.
Cam. Sc. I. Co.	7	Friction drum	H. H. P.	Unpublished.†
Moll (Lick)	± 25	Worm and gear	A. H. F.	<i>L.O.B.</i> , 16, No. 456.

self-standardized by the method of repeated coincidences. The errors in the first three instruments are satisfactorily small and irregular, but in the last two the errors are periodic and unduly large. In a comparison of this kind ruggedness and permanence of adjustment should be taken into account as well as accuracy, and from these points of view the steel band and the friction drum instruments leave perhaps something to be desired. The present instrument appears to rank high both as regards accuracy and sturdiness.

2.3. *Resolving Power*.—A microphotometer of infinite resolving power may be defined as one which would record without distortion the transmission as a function of position on the photographic plate, no matter how the transmission and its successive derivatives with respect to position varied over the plate. Such infinite resolving power would be realized in the Moll-type microphotometer if the slit image on the plate were an infinitely narrow line. On account, however, of the width of the slit itself, and the distribution in the intensity of its image produced by diffraction and aberration, this slit pattern can never correspond to an infinitely narrow line, but only to a more

* H. Coburn, *A.J.*, 42, 75, 1932.

† The usual co-ordinate system was not imposed on these records, so that the error is the sum of errors due to the ways, the screw and the drum.

or less concentrated intensity distribution. It is clear that the more concentrated this slit pattern becomes, the higher will be the resolving power of the microphotometer, so that any practical discussion of resolving power reduces immediately to an observational determination of the slit pattern. Given this, it is possible to find not only a purity or efficiency factor for the instrument, but also the instrumental conditions under which the transmission of a given photographic emulsion will be determined without distortion by the microphotometer.

2.31. *The Slit Pattern.*—In a Moll-type microphotometer let the slit image centred at ξ on the photographic plate (see fig. 5 (a)) have an intensity

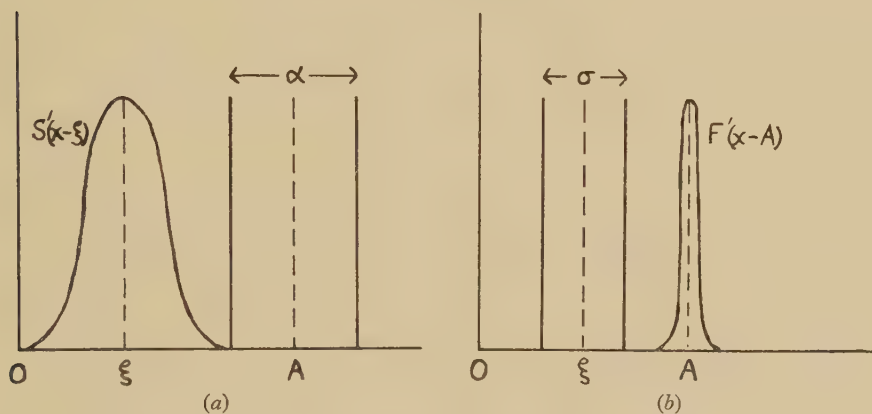


FIG. 5.—Determination of Slit Patterns for (a) Moll- and (b) Koch-type Microphotometers.

distribution $S'(x - \xi)$, where ξ is the centre of the slit pattern with respect to some arbitrary origin on the plate. As the plate is driven across the beam the effect is that the co-ordinate ξ increases uniformly. Now suppose our plate is an Abbe Test Plate consisting of narrow rectangular openings cut in silver on glass, and consider one opening of such a plate centred at A and of width α . Then for the position ξ of the slit the flux which reaches the thermocouple, supposed to be infinitely wide, will be given by

$$I'(\xi) = \int_{A - \frac{\alpha}{2}}^{A + \frac{\alpha}{2}} S'(x - \xi) dx = \int_{A - \frac{\alpha}{2} - \xi}^{A + \frac{\alpha}{2} - \xi} S'(u) du. \quad (1)$$

If, as is more usual, we measure not the flux but the transmission, then the transmission, $I(\xi)$, is given by the ratio of the flux reaching the thermocouple for the position ξ of the slit to the flux when the slit is centred on a large completely transparent part of the plate. Hence we have

$$I(\xi) = I'(\xi) \left/ \int_{-\infty}^{\infty} S'(x) dx \right. = \int_{A - \frac{\alpha}{2} - \xi}^{A + \frac{\alpha}{2} - \xi} S(u) du, \quad (2)$$

where

$$S(x) = S'(x) \left/ \int_{-\infty}^{\infty} S'(x) dx \right.,$$

and consequently

$$\int_{-\infty}^{\infty} S(x) dx = 1. \quad (2a)$$

Now differentiating (2) with respect to ξ we find immediately

$$\frac{dI(\xi)}{d\xi} = S\left(A - \frac{a}{2} - \xi\right) - S\left(A + \frac{a}{2} - \xi\right), \quad (3)$$

or if the centre of the Abbe Test Plate opening be made the origin of co-ordinates (that is, $A = 0$),

$$\frac{dI(\xi)}{d\xi} = S\left[-\left(\xi + \frac{a}{2}\right)\right] - S\left[-\left(\xi - \frac{a}{2}\right)\right]. \quad (4)$$

Thus by a graphical differentiation of the observed transmission curve $I(\xi)$ from an Abbe Test Plate opening of known aperture, it is possible by means of (4) to find the observed slit pattern. Obviously a similar treatment will yield the slit pattern from the transmission curve across a single straight edge.

It is of interest to note that a function $S(x)$, which plays exactly the same part for a Koch-type microphotometer, may also be found from an application of (4). The method of derivation arises from noting that if in the Moll-type microphotometer an infinitely narrow opening is situated at A in fig. 5a, then the flux reaching the thermocouple is $S'(A - \xi)$; provided such an opening could be experimentally realized it would afford an easy method of finding the slit pattern for a Moll-type microphotometer. In fig. 5 (b), which represents the focal plane of the second objective in a Koch-type microphotometer and therefore the plane of the slit, let the Koch slit of width σ be centred at ξ , and let the image centred at A of an infinitely narrow line have an intensity distribution produced by diffraction, aberration, and Schwarzschild-Villiger effect of $F'(x - A)$. As the plate is driven across the analysing beam the co-ordinate ξ uniformly increases, and for any given position ξ the flux which reaches the photo-electric cell situated behind the slit will, from analogy with the Moll-type case, be denoted by $S'(A - \xi)$, where

$$S'(A - \xi) = \int_{\xi - \frac{\sigma}{2}}^{\xi + \frac{\sigma}{2}} F'(x - A) dx. \quad (5)$$

Next suppose that we have an Abbe Test Plate opening centred at A and of width a , and consider an infinitely narrow line in that opening centred at x' . Then from (5) the flux which reaches the photo-electric cell from this line will be

$$S'(x' - \xi) = \int_{\xi - \frac{\sigma}{2}}^{\xi + \frac{\sigma}{2}} F'(x - x') dx,$$

and for the totality of such infinitely narrow lines the flux reaching the photo-electric cell will be

$$I'(\xi) = \int_{A - \frac{a}{2}}^{A + \frac{a}{2}} dx' \int_{\xi - \frac{\sigma}{2}}^{\xi + \frac{\sigma}{2}} F(x - x') dx = \int_{A - \frac{a}{2}}^{A + \frac{a}{2}} S'(x' - \xi) dx'. \quad (6)$$

This is identical with (1), and leads by the same steps to (4). Consequently the slit pattern $S(x)$ can be found from measures of the transmission of an Abbe Test Plate opening by means of (4) for either a Moll- or Koch-type microphotometer. It should be noted, however, that while the Moll-type microphotometer, on account of the linear characteristic curve of its thermocouple, gives this required transmission as a function of ξ directly from the record, with the Koch-type microphotometer it will in general be necessary

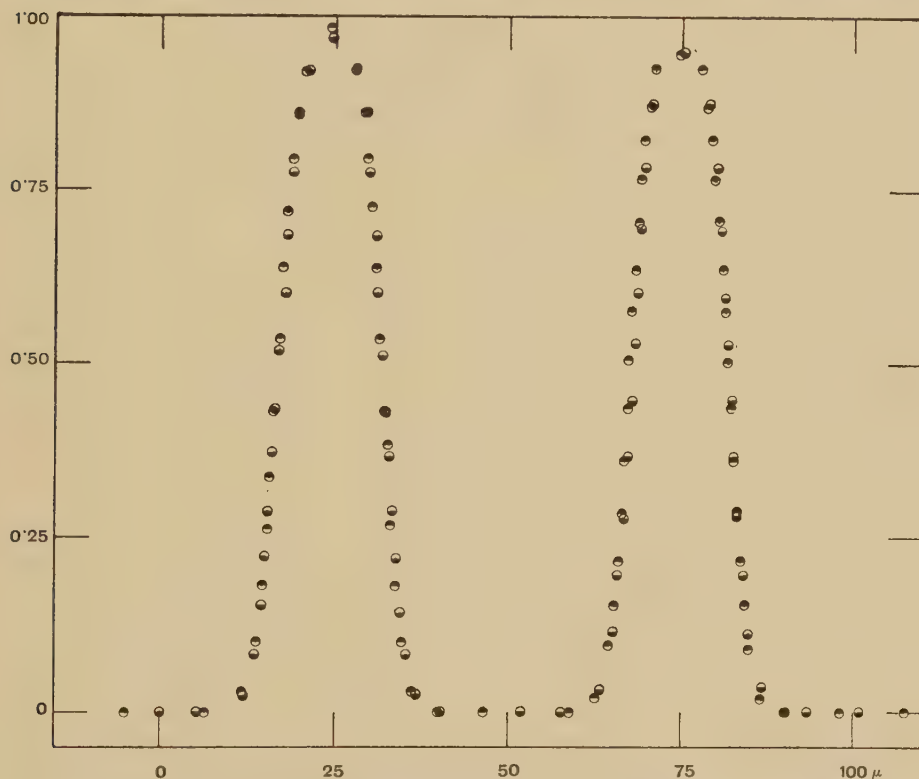


FIG. 6.—Values of $I(\xi)$ for two successive openings of Abbe Test Plate. Circles filled at bottom are for plate travelling from right to left across microphotometer slit image (projected width 5.1μ), and circles filled at top for plate travelling left to right.

first to find the non-linear characteristic curve of its photo-electric cell, and then by its use to change recorded readings to transmissions before an application of equation (4) can be made.

In order to find the slit pattern $S(x)$ for the present instrument two successive openings of an Abbe Test Plate with widths $a = 17.2 \mu$ and $a = 15.4 \mu$, separated by an opaque space of 34.2μ , were run through the microphotometer with magnification 48.2. The measured transmissions, taken from two microphotometer records for opposite directions of run, the x co-ordinates on which were measured on a Hilger micrometer, are shown in fig. 6 for a microphotometer slit width (projected) of 5.1μ . It will be noted that the runs in opposite directions lie well on top of each other, showing the absence of inertia in the thermocouple and galvanometer and

the symmetry in the slit pattern. $I(\xi)$ curves were also obtained for a microphotometer slit width (projected) of 10.4μ .

Placing the origin in turn at the maximum transmission for each $I(\xi)$ curve, each branch of each curve may be graphically differentiated at a number of points to find the complete slit pattern $S(x)$ by means of (4). The resulting slit patterns for the two microphotometer slits used are shown

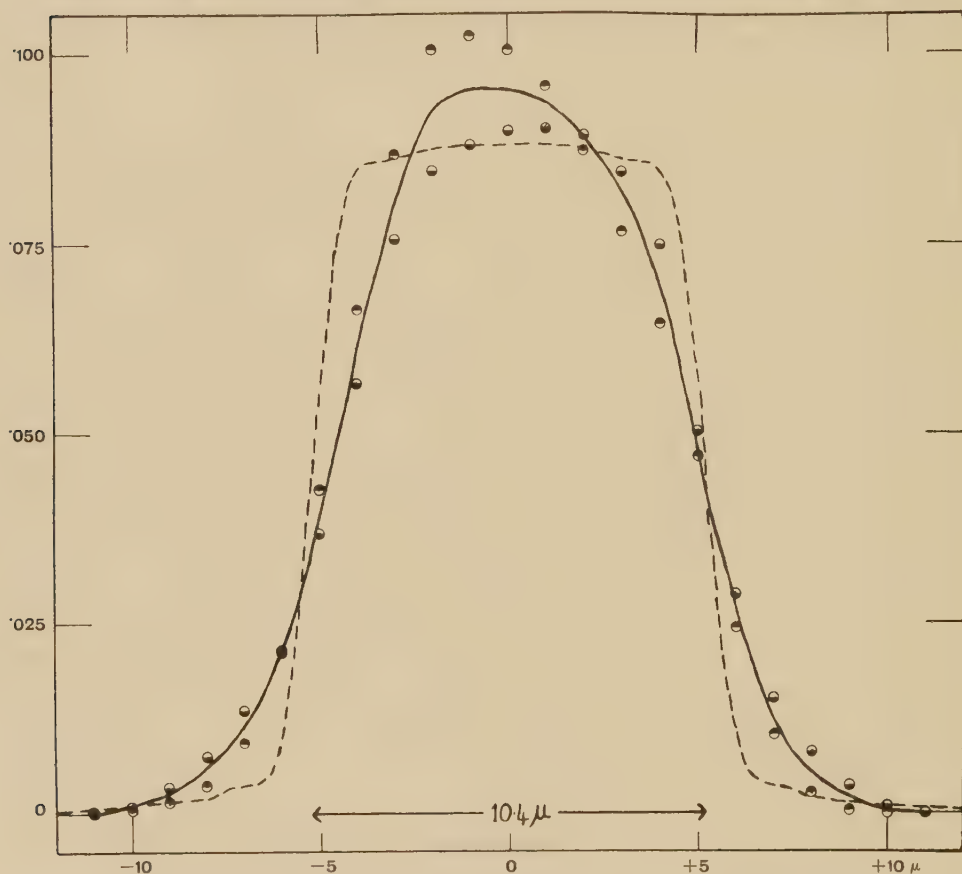


FIG. 7a.—Slit Pattern for Microphotometer Slit 10.4μ . Full line—observed pattern ; broken line—diffraction theory pattern.

as full lines in fig. 7a and 7b, mean observed points from each record being shown by differently half-filled circles. These curves represent the actual distribution of light in the analysing beam as it falls on the plate. The areas of the curves are 0.94 and 0.92 (mean 0.93), and this number, as a moment's consideration will show, is the average transmission of the two Abbe Test Plate openings. If the latter had been transparent, the area under the slit patterns, in accordance with (2a), would have been exactly unity.

2.32. *Purity Factor*.—The theoretical slit pattern for a given microphotometer slit can be found from elementary diffraction theory. Assuming no phase relations between light from different parts of the slit,* as

* Lord Rayleigh, *Scientific Papers*, 4, 235–244, 1896.

illuminated by the lamp-house condenser, it may readily be shown for a microscope objective with rectangular aperture $a \times c$ (a sufficiently close approximation) that the theoretical slit pattern is given by

$$S_t(x) = \frac{ac}{\pi} \int_{v_1}^{v_2} \frac{\sin^2 v}{v^2} dv, \quad (7)$$

where

$$v_1 = \frac{2(\text{N.A.})}{\lambda} \left(x - \frac{\sigma}{2} \right) \pi \quad v_2 = \frac{2(\text{N.A.})}{\lambda} \left(x + \frac{\sigma}{2} \right) \pi.$$

In this equation σ is the projected slit width, (N.A.) is the numerical aperture of the objective (in our case 0.30), and λ is the effective wave-length of the

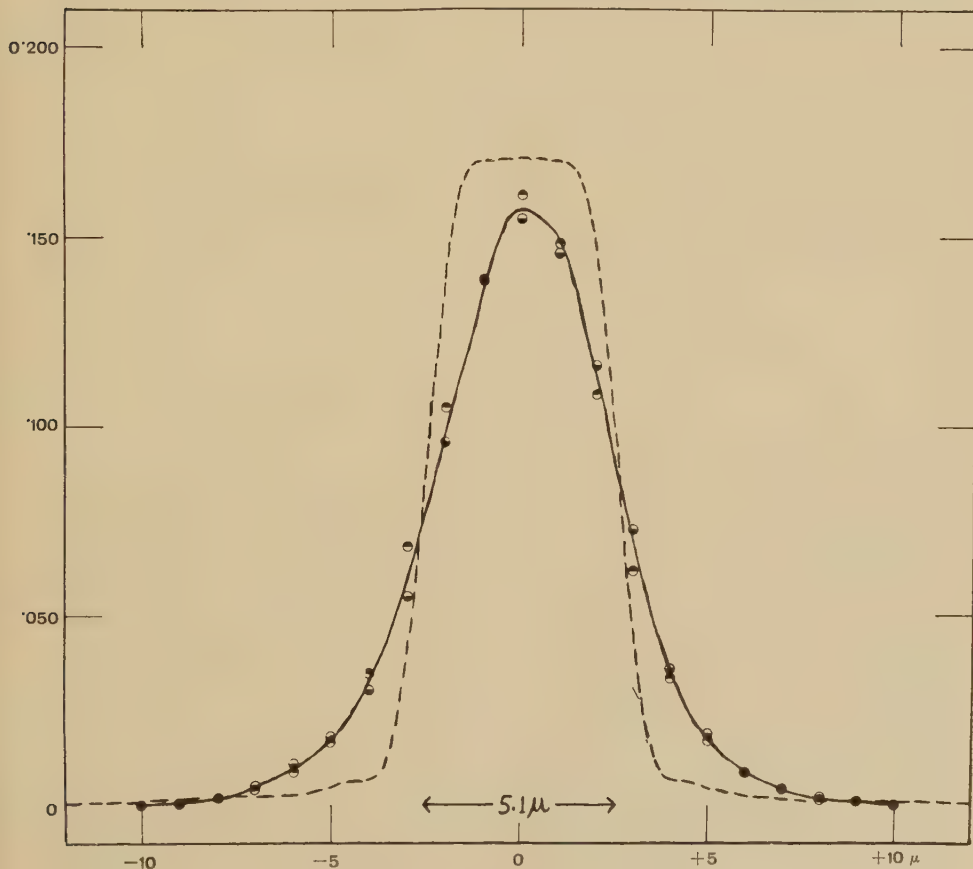


FIG. 7b.—Slit Pattern for Microphotometer Slit 5.1 μ . Full line—observed pattern; broken line—diffraction theory pattern.

analysing beam which may be taken as 1 μ . The value of the integral was found from some new numerical integrations, and the resulting values for $S_t(x)$, with ordinates adjusted so that the area under the theoretical pattern is the same as that for the observed pattern, are shown as broken lines in figs. 7a and 7b.

It will be noted, though the differences are not large, that the theoretical pattern is in each case more concentrated than the observed pattern. This is well shown in Table VI, where the second and third columns contain the widths, for the observed and theoretical patterns respectively, within which the fraction 0.90 of the total light of the pattern lies. The increasing diffuseness of the observed pattern with decreasing slit width, which is shown in this table, is an effect produced by aberration of the optical system of the microphotometer; since the observed patterns are symmetrical within the errors of determination, this aberration presumably arises more from the chromatic difference of focus for $0.56\ \mu$ (the effective wave-length with which the plate is focussed) and $1\ \mu$ (the effective wave-length for which transmission is measured by the microphotometer) than from extra axial aberration originating from lack of collimation.

TABLE VI
Concentration of Slit Patterns

Microph. Slit	0.90 Width		Fraction of Total Light within Projected Slit Width		
	$S(x)$	$S_t(x)$	$S(x)$	$S_t(x)$	Purity
10.4 μ	11.0 μ	9.9 μ	0.879	0.925	0.95
5.1	8.2	5.6	0.720	0.859	0.84

Effects due to aberration must be present to a greater or less extent in all microphotometers, and it is these effects rather than the diffractive widening which impose a limit on the resolving power of the microphotometer. While the material in figs. 7a and 7b is sufficient to evaluate the amount of aberration present in the optical system of the microphotometer, what is needed is rather some simple criterion, applicable to all instruments, which yields a measure of the efficiency of the microphotometer. Probably the simplest of these is found by measuring with a planimeter the fraction of the total light of the observed and theoretical patterns which lies within the limits of the geometrically projected slit width. These fractions are given in the fourth and fifth columns of Table VI. For a given slit width the ratio of these fractions for an observed and theoretical slit pattern may be called the purity for this slit width; this purity factor is given in the last column of the table. For a microphotometer without aberration this purity factor for a given slit width would be unity, but for the same slit width this factor becomes smaller as the aberration makes its appearance and becomes larger. Microphotometers may then be compared as regards their freedom from aberration, or their efficiency as regards resolving power, by determining the purity factor for some standard slit width, say $5\ \mu$. For the present instrument, as reference to Table VI shows, this purity factor is

0.84; it would be of considerable interest to know the purity factors for other microphotometers.

2.33. *Distortion by the Microphotometer.*—Suppose the distribution of transmission in the photographic plate is given by $\mathcal{F}(x)$, and the slit pattern is given by $S(x)$, then for either a Moll- or Koch-type microphotometer it may be shown that $I(\xi)$, the transmission registered by the microphotometer, will be given by

$$I(\xi) = \int_{-\infty}^{\infty} \mathcal{F}(x) \cdot S(x - \xi) dx = \int_{\xi - \beta}^{\xi + \beta} \mathcal{F}(x) \cdot S(x - \xi) dx, \quad (8)$$

the limits of integration being changed, since for all practical purposes a distance β can be found such that $S(x) = 0$ for $-\beta \geq x \geq \beta$. Since we know the slit pattern, $S(x)$, it would be possible to solve the integral equation (8) by successive approximations, and to pass from the measured transmission, $I(\xi)$, to the actual transmission, $\mathcal{F}(x)$, on the plate. In practice, however, such a procedure would be unduly laborious, and what is required is to choose a slit pattern so concentrated that the distortion produced by the microphotometer is negligible. One method of doing this, knowing something of the variation of $\mathcal{F}(x)$ beforehand, is to choose a slit pattern such that over its full extent the value of $\mathcal{F}(x)$ does not differ from its constant value in the middle of the range. In this case the integral equation takes the form

$$I(\xi) = \mathcal{F}(\xi) \int_{\xi - \beta}^{\xi + \beta} S(x - \xi) dx = \mathcal{F}(\xi) \int_{-\beta}^{\beta} S(u) du = \mathcal{F}(\xi) \quad (9)$$

and microphotometer distortion is negligible.

Another method of wider applicability is to choose the slit pattern so that the distortion produced by the microphotometer is less than that already produced by the photographic plate, as a result of the turbidity of its emulsion and the finite size of the silver halide grains. In this case the limit on the accuracy of the transmission curves is that produced by the irremovable limitations of the photographic process, and has nothing to do with the finite resolving power of the microphotometer. In order to find a slit pattern which satisfies this condition it is sufficient to note that the distortion produced by the photographic emulsion is conveniently measured by the so-called photographic resolving power. Imagine a contact print to be made on the photographic emulsion concerned, of a silver on glass plate on which are ruled at regular intervals a number of fine parallel transparent lines; this contact print will show a number of equally spaced black lines. As the spacing of the lines on the silver plate is steadily diminished, a stage is finally reached where, on account of photographic distortion, the emulsion just fails to show separable lines in the contact print. The number of lines per mm. corresponding to this stage is termed the resolving power of the emulsion concerned. Mees, Ross and others at the Kodak Research Laboratory* have carried out a number of investigations on the subject of photographic resolving power, and have been able to show that its determination is independent of the widths of the rulings. The most recent results show

* F. E. Ross, *Physics of the Developed Photographic Image*, chap. iv, 1924.

that the photographic resolving power varies between 35 for fast, coarse-grained emulsions, to 80 for fine-grained process emulsions.* In order now to choose a slit pattern for the microphotometer which will produce less distortion than that already produced by the photographic plate, it suffices to choose a pattern which is adequate to separate more lines per mm. on a similarly ruled silver on glass test plate than the photographic plate can. Consider two such lines, regarded as infinitely narrow, with co-ordinates o and Δ ; then the transmission curve given by the microphotometer will be

$$I(\xi) = S(-\xi) + S(\Delta - \xi). \quad (10)$$

It is a matter of only a few moments numerical trial to use the observed $S(x)$ and equation (10) to find for what value of Δ in microns the transmission curves show a double maximum. The results of such trials for the two slit patterns are given in Table VII, the first and fourth columns containing values of Δ , the second and fifth giving the position of the maxima, and the third and last containing the ratio of $I(\xi)$ at the minimum between the two maxima to $I(\xi)$ at the maxima. Since the microphotometer with slit width

TABLE VII
Microphotometer Resolving Power

Microphotometer Slit 10.4μ			Microphotometer Slit 5.1μ		
Δ	Maxima	$I_{\min}/I_{\max}$	Δ	Maxima	$I_{\min}/I_{\max}$
μ	μ		μ	μ	
9	4.5		4	2	
10	1.5 9.5	0.90	5	0 4	0.95
11	1 11.5	0.69	6	0 6	0.78

10.4μ has a slit pattern sufficiently concentrated clearly to separate 100 lines per mm., it follows that even with this arrangement the distortion produced by the microphotometer is definitely less than that inherent in the fine grain process plates with a maximum resolving power of 80. For faster emulsions a slit with projected width of 20μ would probably give a sufficiently concentrated pattern to give less distortion than that inherent in the emulsion on account of turbidity and graininess. These satisfactory results are due to the small aberration of the optical system, or, otherwise expressed, the high purity factor of the instrument, and the consequence of this is that no existing emulsion would justify the use of the very concentrated pattern given by the slit of projected width 5.1μ .

Summary.—The Moll-type microphotometer described above has certain new mechanical features which make possible accurate collimation and precise plate adjustment. The thermocouple and galvanometer give a linear

* C. E. K. Mees, *Journ. Opt. Soc. America*, **21**, 753, 1931.

characteristic curve which is independent of time. From detailed tests it has been shown that the instrument may be used for the determination of linear displacements with an accuracy of $\pm 2.5 \mu$. Finally, it has been shown that the microphotometer has a high resolving power, so that even with a slit of projected width 10.4μ less distortion is produced by the instrument than is inherent in a fine grain process emulsion—a result due to the freedom of the optical system from aberration.

In conclusion I wish first to acknowledge a special grant from Oxford University which made possible the purchase of this instrument for the observatory, and secondly to express my thanks to R. M. Abraham, to whose ingenuity in design much of the satisfactory performance of the instrument is due.

*University Observatory,
Oxford :
1934 November.*

ON THE CONSTRUCTION OF A SPECTROHELIOSCOPE.

A. M. Newbegin.

It is well known that for many years the work of my observatory has been the observation of solar prominences and the spectrum of sunspots. For this latter work I made a Littrow-type spectroscope of 102 inches focus, fed with a 4-inch grating by Michelson of very fine quality, but the limitations of the usual methods of observation has hitherto prevented seeing prominences anywhere except at the limb of the Sun. When Professor Hale introduced his spectrohelioscope it was at once realised that here was an instrument capable of supplying the deficiencies of the ordinary types of prominence spectroscope, and it was decided to install such an instrument at Worthing as soon as it was practicable. Unfortunately, circumstances hindered that ambition being fulfilled immediately, but ultimately the idea of building the instrument in the workshop suggested itself.

It cannot be claimed that the instrument about to be described embodies any new idea, but it is believed to be the first spectrohelioscope constructed in this country to the designs of Professor Hale and, therefore, some description may be acceptable to other workers in solar physics.

A certain amount of correspondence with Professor Hale, who from the outset evinced great interest in the project, led to his sending me a full set of blue prints of his latest design, which, in the main, have been adhered to, but a few alterations or improvements of a minor character have been introduced.

By the kindness of Sir Frank Dyson, then Astronomer Royal, I was able to examine the Greenwich instrument very thoroughly, and it occurred to me that instrument was somewhat light in construction, and in preparing my patterns I decided to make the main castings heavier with a view to reducing vibration to a minimum.

When the observatory was constructed at Worthing provision was made for the erection of the spectrohelioscope without undue expense, and instead of placing the instrument in a separate observing hut, use has been made of a window and the doorway of the observatory, both having been placed in a north and south line 4 feet west of the meridian of the $6\frac{1}{4}$ -inch equatorial telescope. All that was necessary to house the spectrohelioscope was to build an annexe to the north of the original doorway to take the concave mirrors and make a new entrance on the west side of the annexe. When the dome is closed and the remaining window obscured by a dark blind the observatory is practically in darkness, and there seems no appreciable trouble from atmospheric tremors. On the south side of the observatory a concrete pier carries the cœlostæt, second mirror and object-glass, this part of the instrument being housed in a wood shelter. The cover of this shelter slides right off, leaving the mirrors exposed but protected from the wind by boards 11 inches high which can be lowered as required.

The cœlostæt, second mirror and object-glass are mounted on a cast-iron plate measuring 36 inches $\times$ 20 inches, the V-rails upon which the cœlostæt is moved as required being planed parallel to the length of the plate. This plate, after being adjusted for the meridian, was bedded in cement to ensure the utmost rigidity. The cell carrying the cœlostæt mirror is mounted on steel shafts $\frac{3}{4}$ ths of an inch diameter and rotates in gunmetal bearings. A good clock movement of French make provides the drive to the mirror, the worm-screw being of stainless steel and constructed by Cooke, Troughton & Simms, Ltd., of York, who were also responsible for all the optical parts. A simple clamping device enables the mirror to be rotated independently of the clock.

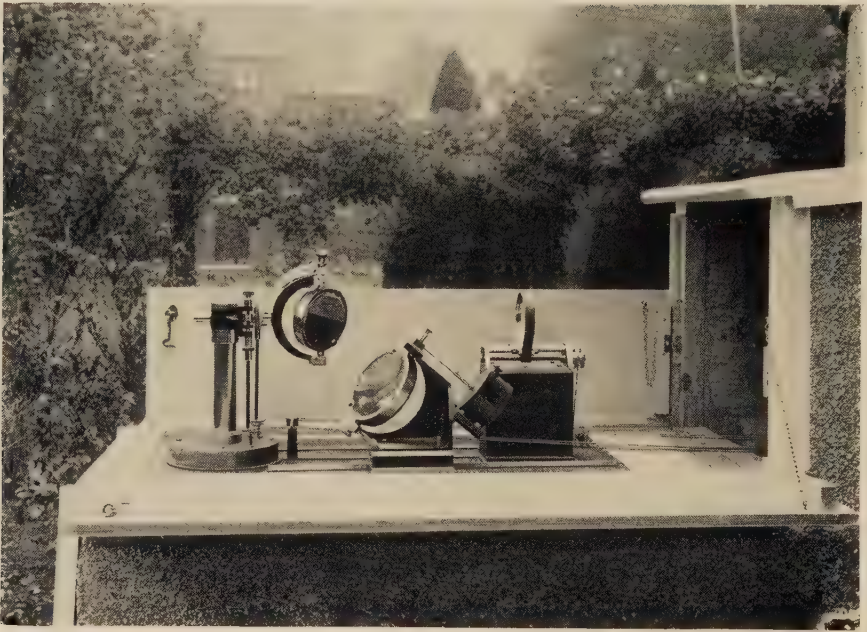
Incidentally, the base plate for the cœlostæt, the clock and the worm-screw were the only parts not made at Worthing. It was realised, in the case of the worm-screw, that great accuracy was essential and this was beyond the capacity of my equipment.

The mirror of the cœlostæt is $5\frac{1}{2}$ inches in diameter and that of the second flat is $4\frac{1}{2}$ inches. Both are made of fused silica and true to about $\frac{1}{4}$ wave. Fused silica is not necessary for such small mirrors and some expense could have been saved by using ordinary plate glass, but there can be no question of permanency of focus with the material adopted. Moreover, it has been stated that fused silica takes the new aluminium coating better than glass, and although in the first instance these mirrors were silvered, when they need recoating they will be treated with aluminium.

The second mirror is mounted on horizontal and vertical axes, both with clamp and slow motions so that it can be rotated or inclined to bring any part of the solar image on to the first slit. The slow motions are worked by cords carried to the eye end of the spectroscope.

The object-glass is a single lens of high transparency boro-silicate crown glass, and is 4 inches clear aperture and 20 feet focus. This is 2 feet longer than in the specification of Professor Hale, and was necessitated by local conditions. It was found that an 18-foot lens would have brought the object-glass close up against the wall of the observatory, and, as it was not possible

(a)



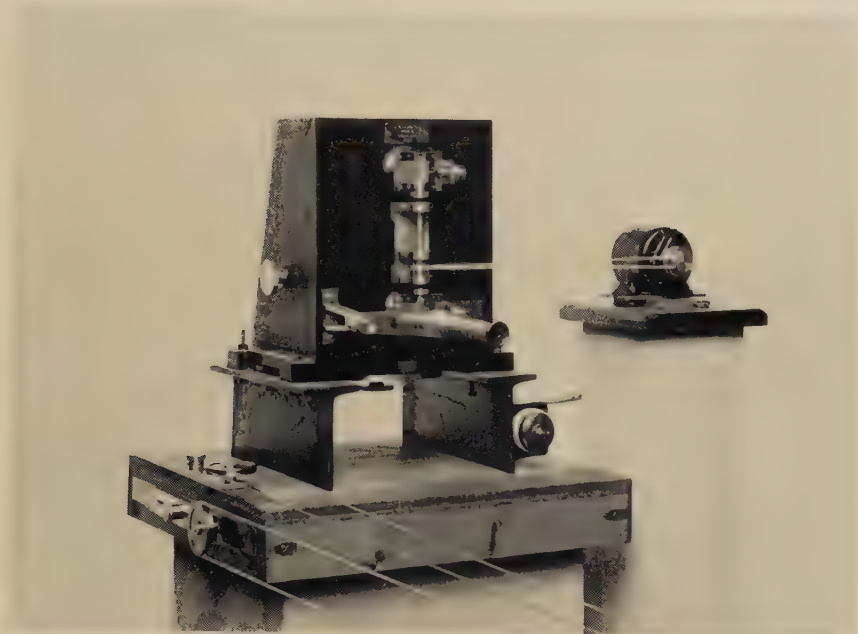
(b)



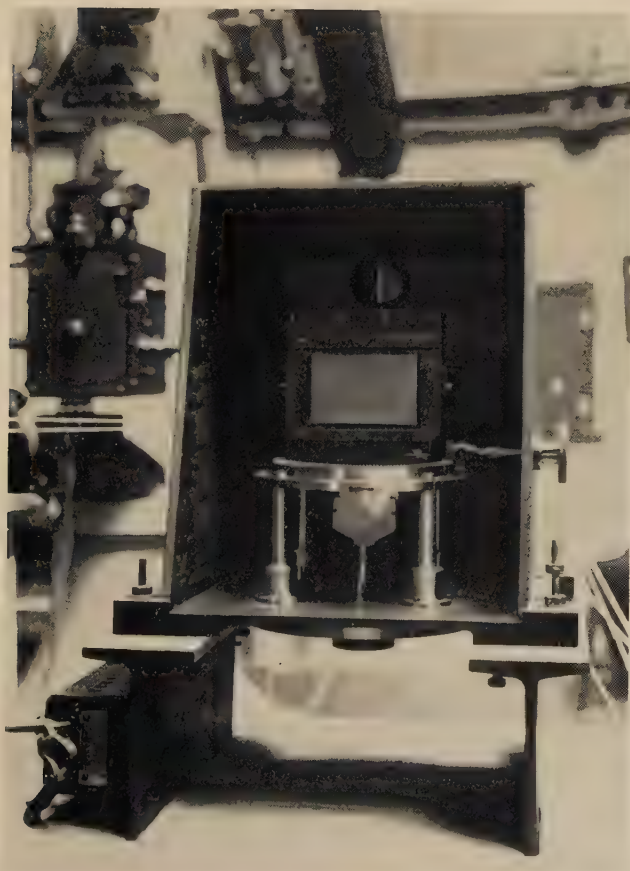
(a) The Cœlostæt, Second Mirror and Object-glass.
(b) The two Concave Mirrors.

A. M. Newbegin, Construction of a Spectroheliograph.

(c)



(d)



(c) Mounting for the Anderson prisms and slits, with the electric motor.
(d) Mounting for the Grating, showing Line-shifter.

A. M. Newbegin, Construction of a Spectroheliometer.

to place the spectroscope farther north, this would probably have been a source of trouble in hot weather.

The cell is mounted on a steel shaft along which it slides and beneath is a lug through which passes a $\frac{3}{8}$ -inch brass screw. This is operated by a cord and pulley from the eye end for purposes of focussing.

The box-casting carrying the slits, Anderson prisms, grating and line-shifter is mounted on two heavy girder pieces, the lower T-piece of each girder being buried in cement to a depth of 4 inches, thus ensuring great stability. These girders are slotted in the top T-pieces to allow of squaring the casting on to the object-glass, and there are also levelling screws.

The slits, each 1 inch in length, are made of stainless steel and ground true with very fine carborundum powder and water on plate glass which had been ground true before starting on the jaws of each slit. Provision is made for lateral adjustment, and the mounts of each slit can be slightly rotated to ensure perfect alignment with the rotating prisms. Each slit is also provided with adjusting screws to ensure the jaws being parallel with each other. The lower slit is further provided with small windows at each end so that the *Ha* line may be brought into coincidence with it. These windows can be occulted when not required by a sliding diaphragm.

The Anderson prisms are $1\frac{9}{16}$ ths inch long and $\frac{3}{8}$ ths inch square. They are made of optical hard crown glass, which is very transparent to the red end of the spectrum, and are true to about $\frac{1}{8}$ wave and the angles true to 5". They are mounted on a steel spindle $\frac{1}{4}$ inch diameter and run in ball bearings, rotation being effected by a small electric motor of $\frac{1}{80}$ h.p., which is controlled by a variable resistance.

Two eye-pieces are provided with powers of $\times 2$ and $\times 10$ respectively. The higher power has an achromatic lens to ensure a flattening of the field.

The grating employed is one which has been in use for a number of years on the Littrow spectroscope for sunspot observations, and the original turntable has been retained. This involved some slight modification of design and it has been mounted very rigidly on four brass columns. The grating is a 4-inch by Michelson and ruled with 15,000 lines to the inch. Rotation is effected by a thumb-screw which operates the geared turntable. The method is somewhat coarse for this class of instrument and some alteration will probably be required.

Beneath the grating is the line-shifter—a piece of plane parallel glass true to $\frac{1}{4}$ wave and measuring $1\frac{1}{2}$ inch $\times$ $1\frac{1}{8}$ inch $\times$ $\frac{1}{16}$ inch. It is connected with a circle underneath the whole casting, the circle being divided up to 70° on either side of zero.

The two concave mirrors, each 3 inches in diameter and 13 feet focus $\pm \frac{1}{4}$ inch, are mounted in cells on a massive gunmetal pillar, the base of which slides in grooves. Focussing is effected by means of a screw and operated by a rod carried to the eye end. Both cells are provided with means of squaring on the mirrors to the optical axis of the whole system, and the lower mirror is further provided with a tilting device about a horizontal axis and controlled by a cord from the eye end. These mirrors are made of fused silica and surfaced with aluminium.

Some preliminary observations have been made and the results indicate the instrument will fulfil the requirements for which it was constructed. Prominences show up well, but for some reason they do not appear so bright as one would expect, and it is possible that a grating specially ruled for the red end of the spectrum is indicated. This is under consideration, but no decision will be taken until all the adjustments have been perfected and the instrument again tried out under favourable atmospheric conditions.

I should like to express my thanks to the Astronomer Royal and Sir Frank Dyson for permission to examine the instrument at Greenwich, to Mr. H. W. Newton for much valuable advice in elucidating the many details of construction, to Mr. James G. Bower for his work in preparing the castings, and lastly to Mr. C. P. Butler for advice in grinding the slits.

NOTE ON SOME OBSERVATIONS OF THE SPECTRA OF THREE B-TYPE STARS IN THE VISUAL REGION.

E. G. Williams, M.A.

(Communicated by the Director of the Solar Physics Observatory)

The plates discussed in this paper were obtained with the four-prism spectrograph on the Newall 25-inch refractor. Both the D-line and $H\alpha$ regions were in good focus, and the scale on the plates at $\lambda\lambda$ 5875, 6563 was 34 and 52 A./mm. respectively. The starlight was brought to a focus on the slit in a wave-length slightly to the red of $H\alpha$. Thus not only did the rapid increase of intensity with decreasing wave-length in early type stars enable spectra to be obtained which were suitably dense at both $H\alpha$ and in the D-line region, but the intensity of the continuous spectrum on the plate was approximately the same on both sides of $H\alpha$, an important consideration when contours are to be measured for lines with extensive wings.

The plates were calibrated for photometric use in an auxiliary spectrograph fitted with a neutral-tint wedge, and were analysed with the recording microphotometer.

Table I gives particulars of all the plates measured for line intensity. The total absorptions are expressed as the equivalent width of the lines in units of 0.01 angstrom, and the central depths, enclosed in brackets, are percentages. The letters I, E after the plate number refer to the emulsions: Ilford Hypersensitive Panchromatic, and Eastman III F. The latter type, though slower than the Ilford plates, had finer grain and gave greater contrast. All plates were ammoniated before use.

Hydrogen.—Table II collects the data of Table I for hydrogen into weighted mean values, and includes in addition numerical results for other lines obtained by the writer at Mt. Wilson,* by Elvey† and by Günther.‡

* *P.A.S.P.*, 46, 292, 1934.

† *Ap. J.*, 71, 191, 1930.

‡ *Zs. für Astrophys.*, 7, 106, 1933.

TABLE I
Measures of Total Line Absorption

Star	Plate	<i>H</i> α	Wt.	λ 5876 <i>He</i>	Wt.	λ 6678 <i>He</i>	Wt.
α Leonis	FR 7 E	740 (39)	3	15 (2.5)	2	16 (5)	2
	11 I	650 (37)	1				
η Ursæ Majoris	FR 5 E	670 (36)	3	60 (12)	1	93 (11)	2
	6 E	590 (35)	3	50 (13)	1	60 (11)	1
	14 E	580 (35)	2	65 (12)	3	66 (11)	1
ζ Persei	FR 3 E	200 (32)	2	300 (75)	3	137 (28)	3

In forming the means given in the last line for each star in Table II, each entry has been given unit weight, except the present measures for *H* α , which were assigned weight 2. The average deviation from the mean is 12 per cent. for *H* α and 6 per cent. for the other lines.

TABLE II
Collected Results for the Total Intensities and Central Depths of the Hydrogen Lines

Star	Observer	<i>H</i> α	<i>H</i> β	<i>H</i> γ	<i>H</i> δ	<i>H</i> ϵ	<i>H</i> ζ
α Leo	Williams	720 (39)	700 (48)	870 (55)	730 (58)	900 (60)	760 (60)
	Günther	650	820 (54)	870 (63)	810 (65)	750	710
	Elvey	920	630 (45)	950 (55)	880		
	Mean	753	717	897	807	825	735 *
η UMa	Williams	620 (35)	785 (48)	746 (53)	780 (56)	842 (57)	708 (56)
	Günther	700	620 (46)	800 (57)	760 (60)	690	610
	Elvey	520	660	850	690		
	Mean	615	688	799	743	766	660 *
ζ Per	Williams	200 (32)	196 (39)	218 (52)	220 (50)	205 (49)	201 (47)
	Günther	...	248 (37)	187 (46)	210 (48)	223	163
	Elvey	...	218 (46)	250 (60)			
	Mean	200	221	218	215	214	182

* These low values are probably partly due to slight overlapping of the wings with *H* η .

The lines $H\alpha$ to $H\zeta$ behave similarly in α Leonis, B8n, and η Ursæ Majoris, B3n. The tendency is for $H\alpha$ to be the weakest line, followed by $H\beta$, whilst $H\gamma$ is the strongest. In all three stars the central depths increase substantially from $H\alpha$ through $H\beta$ to $H\gamma$, after which there is but slight variation.

In a recent paper * comparing the Paschen and Balmer lines Merrill and Wilson found that the contours in the case of the stars α Leonis, α Cygni and β Orionis are such that the widths of lines in the two series at depths which are equal fractions of the respective central depths are almost exactly proportional to the wave-length. It is interesting to note that this proportionality of line width to wave-length holds approximately for the lines in Table II. To demonstrate this the total absorptions were converted into their equivalent widths in micro-wave lengths,† $10^{-6} \lambda$; that is, the area was expressed in units proportional to the wave-length. Then these converted areas were reduced to the values they would have if the central depths were all equal. This procedure yields the following relative widths :—

	$H\alpha$	$H\beta$	$H\gamma$	$H\delta$	$H\epsilon$	$H\zeta$
α Leo	0.89	0.94	1.14	1.03	1.06	0.96
η UMa	0.86	0.94	1.14	1.03	1.08	0.97
ζ Per	0.91	1.12	0.92	1.01	1.06	0.96

The figures for the first two stars agree surprisingly well and show that $H\alpha$ is definitely narrower than $H\gamma$. In the case of ζ Persei, B1s, the $H\alpha$ figure is based on one plate only, and the proportionality to wave-length holds within the limits of observational error. Some attention must be given to the influence of dispersion on line depth before final conclusions can be reached. The dispersions at $H\alpha$, $H\beta$, $H\gamma$ used in deriving the above result were in the ratios 5 : 4 : 7.

TABLE III

Collected Results for the Equivalent Widths and the Depths of Helium Lines

	$2^3P^0 - m^3D$				$2^1P^0 - m^1D$					
	5875	4471	4026	3820	6678	4922	4388	4144	4009	3927
α Leo	15 (3)	36 (10)	40 (9)	25 (6)	16 (5)	37 (9)	37 (6)	39 (5)	(3)	
η UMa	61 (13)	128 (24)	162 (30)	111 (25)	74 (11)	74 (14)	82 (16)	64 (15)	86 (13)	76 (8)
ζ Per *	300 (75)	114 (41)	95 (43)	94 (37)	137 (28)	76 (22)	64 (35)	50 (25)	37 (22)	39 (20)

* Elvey (*Ap. J.*, 70, 141, 1929) finds for the depth of 4471, 46, and for 4388, 34.

* *Mt. Wilson Contr.*, No. 494; *Ap. J.*, 80, 31, 1934.

† Dunham, *Publ. Amer. Astr. Soc.*, 7, 215, 1933.

Helium.—Table III summarizes the results for the helium lines. The measures in the ordinary photographic region were made on spectra secured by the writer at Mt. Wilson. In α Leonis and η Ursæ Majoris helium intensity attains a maximum at about the third line in both the diffuse triplet and diffuse singlet series. This is best seen from examination of the depths, which are more reliable than the areas on account of the uncertainty of measurement of extreme widths for shallow lines. The variation with series number is more pronounced for the triplets than for the singlets, and the decline seems equally rapid on either side of the strongest line.

In contrast to this behaviour we find for ζ Persei a tolerable uniformity of intensity and contour for the lines in the blue-violet region, whilst the two leading lines of their respective series are greatly enhanced in width and, in the case of the triplets, in depth. Though this result is based on only one plate there is no evident reason for suspecting the relative strength of these two lines, and it is suggested that the phenomenon is associated with luminosity, for it is probable that ζ Persei is three or four magnitudes brighter than the other two stars. All lines in ζ Persei are sharp, whilst those in α Leonis and η Ursæ Majoris are very diffuse. On plates of the D-line region, taken here by Mr. Thackeray, the line 5875 *He* in β Orionis, with total absorption 58 and depth 23, is somewhat stronger than the other diffuse triplet lines in this star. In α Cygni D₃ appears definitely stronger than both 4471 and 4026.

Peculiarities in H α .—On F R 7, a plate of α Leonis of very good quality at *H α* , two rather sharp absorption lines are clearly discernible on the more refrangible wing of that line, presenting the appearance of satellites. The wave-lengths, corrected for radial velocity, are 6547.91, 6552.70. Other plates of *H α* in α Leonis have not enough contrast to show these weak lines, and they are recorded neither in α Cygni * nor by Miss Heger † in B-type stars. It is possible that they may be identified with the atmospheric lines $\lambda\lambda$ 6547.7, 6548.6, 6552.6, of which the intensities given in the Revised Rowland Table are 0, 1, 1 respectively, but neither of the air lines $\lambda\lambda$ 6533.9, 6543.9, both of intensity 2, can be seen in α Leonis. Nor are any of these atmospheric lines visible on the plate of ζ Persei, though this star was observed at a lower altitude in the sky than was α Leonis.

In η Ursæ Majoris no structure can be detected in the wing of the line, but instead, on F R 5, which is the best plate available, a very faint narrow emission line is strongly suspected at the line centre. The plate was taken on 1934 May 4.

It is a pleasure to express my thanks to Professor Stratton for the facilities he has given me, and for his generous assistance in securing observations.

Cambridge :
1934 December 6.

* Waterman, *L.O.B.*, 8, 1, 1913 ; Wright, *ibid.*, 10, 108, 1921.

† *L.O.B.*, 10, 146, 1922.

MICROMETRICAL MEASURES OF DOUBLE STARS.

Rev. T. E. R. Phillips, M.A.

The following measures have been made with the 8-inch refractor lent by the Society:—

Special attention has been given to Σ 1523 (ξ UMa), and in view of the extremely rapid motion of the *comes* in the neighbourhood of periastron, and the apparent irregularities due to the invisible companion, means of the measured positions are given at less than the yearly intervals as well as annual values.

Star	1925		Mags.	Position Angle	Dis- tance	No. of Nights	Date 1900 +
	R.A.	Dec.					
	^h ^m	[°] [']					
AC 1	0 16.9	+32 34	6.5, 7.0	286.7	1.43	3	33.88
OΣ 12 (λ Cas) ^a .	0 27.6	+54 7	5.6, 5.9	162.9	0.43	3	31.72
Σ 60 (η Cas)	0 44.4	+57 26	4.0, 7.6	273.0	8.46	4	31.69
				273.7	8.61	3	33.57
				275.1	8.69	4	34.71
Σ 73 (36 And)	0 50.9	+23 14	6.2, 6.8	85.4	0.54	7	31.47
				89.8	0.52	7	32.84
Σ 113 (42 Cet)	1 16.0	- 0 54	6.2, 7.2	1.9	1.52	4	31.74
Σ 174 (1 Ari)	1 46.0	+21 54	6.2, 7.4	166.4	2.82	3	31.77
				165.3	2.75	3	34.70
Σ 180 (γ Ari)	1 49.4	+18 55	4.2, 4.4	359.7	8.04	3	31.71
Σ 186	1 52.0	+ 1 28	7.2, 7.2	44.8	1.14	5	31.73
Σ 202 (α Psc)	1 58.2	+ 2 24	2.8, 3.9	308.7	2.48	4	31.72
OΣ 38 (γ^2 And)	1 59.3	+41 58	5.0, 6.2	105.3	0.49	7	31.80
				105.5	0.52	6	33.70
Σ 285	2 34.4	+33 6	7.0, 7.7	168.1	1.66	3	31.85
Σ 299 (γ Cet)	2 39.4	+ 2 56	3.7, 6.2	293.0	2.79	3	33.88
Σ 300	2 40.2	+29 9	7.9, 8.1	307.8	3.13	4	33.90
Σ 305	2 43.2	+19 4	7.3, 8.2	314.0	3.48	5	31.80
OΣ 95	5 1.1	+19 42	7.0, 7.6	316.5	0.84	3	34.00
OΣ 98	5 3.8	+ 8 24	5.9, 6.7	115.4	0.88	3	33.95
Dawes 5 (η Ori)	5 20.7	- 2 27	3.7, 4.9	79.7	1.27	3	33.93
Σ 738 (λ Ori)	5 31.0	+ 9 53	3.7, 5.6	42.7	4.24	3	33.97
Σ 795	5 44.0	+ 6 26	6.0, 6.0	207.4	1.36	3	33.98
Σ 1110 (Castor)	7 29.8	+32 3	2.0, 2.8	208.8	4.21	4	31.29
				207.3	4.03	10	32.97
Σ 1196 (ζ Cnc) ^b .	8 7.9	+17 53	5.6, 6.3	169.3	0.50	3	31.34
				149.5	0.54	3	32.96
				135.4	0.62	4	34.12
Σ 1223	8 22.3	+27 11	6.3, 6.3	216.1	5.03	2	34.10
Σ 1355	9 23.4	+ 6 34	7.5, 7.5	158.4	2.68	2	34.12
Σ 1356 (ω Leo)	9 24.4	+ 9 23	5.9, 6.7	139.7	0.89	3	34.08
OΣ 215	10 12.2	+18 6	7.2, 7.4	194.9	1.11	3	34.18
Σ 1424 (γ Leo)	10 15.8	+20 13	2.6, 3.6	118.2	3.87	3	31.29
				118.6	4.00	3	34.13

Star	1925				Mags.	Position Angle	Dis- tance	No. of Nights	Date 1900 +
	R.A.		Dec.						
	^h 11	^m 14.2	[°] +31	['] 58					
Σ 1523 (ξ UMa)					4.0, 4.9	°	"		
Annual Means	{	{	37.3	1.04	1931.41	39.4	1.11	8	31.28
						35.8	0.99	12	31.50
						32.6	0.93	3	31.82
			23.1	0.86	1932.27	24.3	0.92	3	32.21
						15.0	0.75	4	32.51
						1.1	0.87	4	33.01
			353.5	0.89	1933.20	351.9	0.97	4	33.40
						347.5	0.83	4	33.50
						333.6	0.96	5	34.03
			329.6	1.02	1934.34	325.5	1.06	8	34.53
Σ 1536 (ι Leo)	11	20.0	+10	57	3.9, 7.1	15.2	0.94	3	31.32
Σ 1670 (γ Vir)	12	37.9	- 1	2	3.6, 3.7	318.9	5.80	6	31.27
Σ 1687 (35 Com)	12	49.6	+21	39	5.0, 7.8	109.2	0.94	3	31.34
Σ 1865 (ζ Boo)	14	37.6	+14	3	4.4, 4.8	133.7	1.05	6	31.33
						132.4	1.03	7	33.52
						132.8	1.06	5	34.55
Σ 1877 (ε Boo)	14	41.7	+27	23	3.0, 6.3	333.6	2.80	5	31.40
Σ 1888 (ξ Boo)	14	47.9	+19	24	4.7, 6.6	23.4	4.01	6	31.37
						19.0	4.20	8	33.44
						18.6	4.40	4	34.52
Σ 1937 (η CrB)	15	20.1	+30	33	5.2, 5.7	204.0	0.56	10	31.43
						220.6	0.54	8	33.53
						234.8	0.50	4	34.57
Σ 2055 (λ Oph) ^e .	16	27.1	+ 2	8	4.0, 6.1	121.8	0.43	3	31.47
Σ 2114	16	58.4	+ 8	33	6.6, 7.8	170.9	1.20	3	33.60
Σ 2140 (α Her)	17	11.2	+14	28	3.5, 5.5	109.6	4.62	5	31.45
OΣ 338	17	48.6	+15	20	7.2, 7.4	5.7	0.76	3	33.60
Σ 2244	17	53.2	+ 0	4	6.8, 7.0	278.2	0.64	3	33.60
Σ 2264	17	58.3	+21	36	5.1, 5.2	259.0	6.31	6	31.70
Σ 2262 (τ Oph)	17	59.0	- 8	11	5.3, 6.0	265.3	2.06	3	33.63
Σ 2272 (70 Oph)	18	1.7	+ 2	33	4.3, 6.0	121.8	6.72	4	31.47
						120.4	6.77	5	33.55
						119.2	6.62	3	34.57
Σ 2289	18	6.8	+16	27	6.5, 7.6	225.0	1.13	3	33.60
OΣ 358	18	32.5	+16	55	6.7, 7.1	181.0	1.89	3	33.60
Σ 2375	18	41.8	+ 5	25	6.2, 6.6	115.1	2.60	3	31.51
						115.7	2.58	3	33.60
Σ 2382 (ε ¹ Lyr)	18	41.9	+39	35	5.1, 6.1	5.6	2.94	4	31.54
Σ 2383 (ε ² Lyr)	18	41.9	+39	32	5.1, 5.4	114.0	2.27	4	31.54
Σ 2404	18	45.1	+10	50	5.8, 7.0	181.8	3.49	3	31.49
Σ 2451	19	1.2	+51	29	8.7, 9.0	68.9	2.16	3	33.79
Σ 2525	19	23.5	+27	10	8.4, 8.6	300.5	1.24	4	33.54
Σ 2579 (δ Cyg)	19	42.6	+44	56	3.0, 7.7	270.4	1.79	8	31.57
						267.1	1.92	4	33.59
Σ 2596	19	50.6	+15	6	7.3, 8.7	320.5	1.86	3	33.77
OΣ 395 (16 Vul)	19	58.8	+24	43	5.9, 6.3	107.9	0.82	3	33.55
Σ 2644	20	8.8	+ 0	39	6.8, 7.1	27.6	3.01	3	33.65
OΣ 410	20	36.8	+40	18	6.5, 6.8	12.4	0.65	3	33.68
Σ 2716	20	38.0	+32	3	5.9, 8.0	47.8	2.71	4	33.68
Σ 2723	20	41.3	+12	3	6.9, 8.7	106.4	1.12	3	33.67
OΣ 413 (λ Cyg)	20	44.5	+36	13	4.8, 6.1	38.7	0.68	3	31.54
						40.0	0.60	3	33.67

Star	1925				Mags.	Position Angle	Dis- tance	No. of Nights	Date 1900 +
	R.A.		Dec.						
	^h	^m	[°]	[']					
Σ 2735	20	51.9	+ 4	14	6.3, 7.8	284.5	2.06	4	33.65
Σ 2742	20	58.5	+ 6	54	7.4, 7.4	218.9	2.80	3	33.65
Σ 2751	21	0.1	+56	23	6.1, 7.1	351.4	1.77	3	33.67
Σ 2762	21	5.5	+29	54	5.7, 7.7	128.6	3.53	3	34.70
Σ 2780	21	9.9	+59	41	6.0, 7.0	218.8	1.14	4	33.68
OΣ 437	21	17.7	+32	8	6.9, 7.6	31.8	2.07	3	33.55
Σ 2799	21	25.2	+10	46	7.4, 7.4	285.3	1.65	7	31.59
Σ 2822 (μ Cyg)	21	40.8	+28	30	4.7, 6.1	164.8	0.66	3	31.57
						176.1	0.60	4	33.62
Σ 2878	22	10.8	+ 7	36	6.8, 8.3	123.2	1.31	5	34.70
Σ 2881	22	11.2	+29	12	7.6, 8.1	91.0	1.47	4	34.70
Σ 2909 (ζ Aqr)	22	25.0	- 0	24	4.4, 4.6	297.5	2.52	10	31.71
Σ 2944	22	44.0	- 4	37	7.3, 7.8	267.5	2.92	2	33.72
A.G.C. 14	23	40.2	+28	58	5.0, 8.1	212.9	0.91	2	31.41
Σ 3042	23	48.1	+37	29	7.8, 7.8	87.4	5.20	3	31.80
Σ 3050 (37 And)	23	55.7	+33	19	6.6, 6.6	237.3	1.85	3	31.76

Notes

a. Distance as given probably slightly too small; but the star was apparently notched only—scarcely divided.

b. Widening again. Minimum distance apparently about 1929.

c. Star notched only—not divided.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE UNIVERSITY OF LONDON OBSERVATORY, BETWEEN 1933 NOVEMBER 3 AND 1934 OCTOBER 1.

(Communicated by the Director)

This report concerning occultations observed and reduced at Mill Hill is the fourth of a series of which No. 3 is in *M.N.*, 94, 91, 1933. "Eye and ear" is the method of observation adopted by G and sometimes by R, who relies chiefly on a stop-watch. It may not be known generally that the use of the stop-watch has been recommended by Professor E. W. Brown (*J.B.A.A.*, 37, 96, 1926).

The three co-ordinates of the 24-inch Wilson reflector are given year by year in the *N.A.*; as a Newtonian its focal length is 126 inches, increased as a Cassegrain to 532 inches. The 4-inch and the 6-inch refractors are respectively the finder and the guider for the reflector. The 8-inch Cooke is under a separate dome at 17 metres distance (considered negligible in the reductions). The agreement between the two observers is satisfactory whenever they observed the same phenomenon. Column 6 of Table I contains the actual G.C.T. of the observations. For the purpose of interpolating for parallax, right ascension and declination (but for no other pur-

pose) $0^h.00253$ was added to the G.C.T. to correct, for the known error in the mean longitude, as desired by Professor Brown. No other correction for lag, personal error, known irregularities in the Moon's limb, etc., has been included.

The reductions were made by R and checked by G.

TABLE I

Serial No.	Star	Mag.	Phase	Date	G.C.T.	Telescope	Observer
				1933	h m s		
39	Taygeta	4.3	D	Nov. 3	20 15 39.84	R	R
40	Maia	4.1	D	3	20 21 6.19	R	R
41	Asterope 2	7.0	D	3	20 40 18.81	R	R
42	Asterope 1	7.0	D	3	20 41 37.59	R	R
43	B.D. $24^\circ 562$	5.4	R	3	20 54 8.54	R	R
44	Taygeta	4.3	R	3	21 7 0.43	R	R
45	Maia	4.1	R	3	21 27 22.12	R	R
46	μ Cancr	5.5	R	Dec. 6	5 55 40.56	R	R
				1934			
47	47 B Ari	6.5	D	Jan. 22	17 59 41.75	R	R
48	7 Tau	5.9	D	Feb. 20	19 20 38.23	R	R
49	B.D. $26^\circ 1082$	7.0	D	Mar. 23	0 20 9.52	N	G
50	A Leo	4.6	D	27	20 32 12.86	G	G
51	A Leo	4.6	D	27	20 32 13.21	R	R †
52	A Leo	4.6	D	27	20 32 13.63	R	R
53	48 Gem	5.8	D	May 17	22 7 7.08	G	G
54	p^4 Leo	5.6	D	22	23 19 40.90	R	R
55	ϕ Sag	3.3	D	31	1 52 9.39	G	G
56	ϕ Sag	3.3	R	31	2 40 3.40	G	G
57	17 Tau	3.8	D	Aug. 31	0 58 12.68	R	R
58	17 Tau	3.8	D	31	0 58 18.26	G	G
59	q Tau	4.3	D	31	1 19 8.24	R	R †
60	q Tau	4.3	D	31	1 19 9.24	G	G
61	q Tau	4.3	D	31	1 19 9.52	R	R
62	20 Tau	4.1	D	31	1 26 55.07	R	R
63	20 Tau	4.1	D	31	1 26 55.25	G	G
64	17 Tau	3.8	R	31	1 47 53.11	F	G
65	16 Tau	5.4	R	31	2 2 40.98	R	R
66	Anon.	?	R	31	2 5 42.08	F	G
67	q Tau	4.3	R	31	2 17 27.15	R	R
68	20 Tau	4.1	R	31	2 36 43.97	F	G
69	20 Tau	4.1	R	31	2 36 44.14	R	R
70	21 Tau	5.8	R	31	2 39 41.49	R	R †
71	21 Tau	5.8	R	31	2 39 40.99	R	R
72	22 Tau	6.5	R	31	2 46 11.41	R	R
73	W.Z.C. 227	6.6	R	31	3 39 43.11	G	G
74	B.D. $26^\circ 884$	6.5	R	31	4 14 41.98	R	R †
75	B.D. $26^\circ 884$	6.5	R	31	4 14 42.65	R	R
76	26 Ari	6.2	R	Sept. 26	3 45 49.63	R	R †
77	26 Ari	6.2	R	26	3 45 50.19	R	R
78	W.Z.C. 350	6.8	R	29	0 2 37.75	R	R †
79	W.Z.C. 350	6.8	R	29	0 2 37.90	C	G
80	W.Z.C. 350	6.8	R	29	0 2 38.71	R	R
81	ω Gem	5.2	R	Oct. 1	1 1 5.92	R	R

TABLE II

Serial No.	χ	ρ	$\chi - \rho$	$\sigma' - \sigma$	$\cos (\chi - \rho)$	$\sin (\chi - \rho)$
39	$30^{\circ} 15'$	$76^{\circ} 52'$	$313^{\circ} 23'$	$+3.2''$	$+ .687$	$-.727$
40	$65 47$	$76 53$	$348 54$	-0.1	$+ .981$	$-.193$
41	$33 31$	$76 58$	$316 33$	$+0.3$	$+ .726$	$-.688$
42	$22 42$	$76 58$	$305 44$	$+2.4$	$+ .584$	$-.812$
43	$253 52$	$77 1$	$178 51$	$+3.0$	-1.000	$+ .020$
44	$286 57$	$77 3$	$209 54$	$+2.2$	$- .867$	$-.498$
45	$251 17$	$77 8$	$174 9$	$+0.9$	$- .995$	$+ .102$
46	$274 59$	$106 44$	$168 15$	-1.5	$- .979$	$+ .204$
47	$22 29$	$68 39$	$313 50$	-3.2	$+ .693$	$-.721$
48	$24 5$	$76 41$	$307 24$	-1.5	$+ .607$	$-.794$
49	$70 16$	$93 58$	$336 18$	-1.0	$+ .916$	$-.402$
50	$106 20$	$115 22$	$350 58$	$+0.5$	$+ .988$	$-.157$
51	$106 20$	$115 22$	$350 58$	$+0.3$	$+ .988$	$-.157$
52	$106 20$	$115 22$	$350 58$	$+0.2$	$+ .988$	$-.157$
53	$66 39$	$101 21$	$325 18$	± 0.0	$+ .822$	$-.569$
54	$66 21$	$117 0$	$309 21$	$+0.3$	$+ .634$	$-.773$
55	$126 53$	$81 45$	$45 8$	$+2.3$	$+ .705$	$+ .709$
56	$209 50$	$81 30$	$128 20$	$+1.4$	$- .620$	$+ .784$
57	$111 58$	$78 11$	$33 47$	$+0.9$	$+ .831$	$+ .556$
58	$112 4$	$78 11$	$33 53$	-1.0	$+ .830$	$+ .558$
59	$38 0$	$78 16$	$319 44$	-2.2	$+ .763$	$-.646$
60	$37 59$	$78 16$	$319 43$	-2.5	$+ .763$	$-.647$
61	$37 58$	$78 16$	$319 42$	-2.6	$+ .763$	$-.647$
62	$71 14$	$78 17$	$352 57$	-1.0	$+ .992$	$-.123$
63	$71 15$	$78 17$	$352 58$	-1.0	$+ .992$	$-.122$
64	$208 10$	$78 22$	$129 48$	$+2.0$	$- .640$	$+ .768$
65	$249 38$	$78 26$	$171 12$	-2.6	$- .988$	$+ .153$
66	Star awaiting identification					
67	$281 44$	$78 29$	$203 15$	$+1.3$	$- .919$	$-.395$
68	$248 50$	$78 34$	$170 16$	$+2.4$	$- .986$	$+ .169$
69	$248 50$	$78 34$	$170 16$	$+2.4$	$- .986$	$+ .169$
70	$289 37$	$78 35$	$211 2$	$+2.3$	$- .857$	$-.516$
71	$289 37$	$78 35$	$211 2$	$+2.2$	$- .857$	$-.516$
72	$278 46$	$78 36$	$200 10$	$+2.1$	$- .939$	$-.345$
73	$189 15$	$78 49$	$110 26$	-1.3	$- .349$	$+ .937$
74	$257 29$	$91 7$	$166 22$	$+2.1$	$- .972$	$+ .236$
75	$257 29$	$91 7$	$166 22$	$+2.3$	$- .972$	$+ .236$
76	$267 42$	$71 42$	$196 0$	$+4.1$	$- .961$	$-.276$
77	$267 42$	$71 42$	$196 0$	$+4.4$	$- .961$	$-.276$
78	$290 29$	$88 1$	$202 28$	$+1.4$	$- .924$	$-.382$
79	$290 28$	$88 1$	$202 27$	$+1.6$	$- .924$	$-.382$
80	$290 28$	$88 1$	$202 27$	$+1.9$	$- .924$	$-.382$
81	$241 51$	$100 14$	$141 37$	$+1.1$	$- .784$	$+ .621$

TABLE III

Serial No.	Star	R.A. 1933.0			Reduction to Date	Dec. 1933.0			Reduction to Date	Authority
		h	m	s		°	'	"		
41	$l=22$ Tau	3	42	3.12	+4.87	+24	19	13.0	+21.5	N.A. 1933
42	$\kappa=21$ Tau	3	41	54.70	+4.87	+24	20	48.3	+21.5	N.A. 1933
		R.A. 1934.0				Dec. 1934.0				
49	B.D. 26° 1082	6	3	12.85	+1.67	+26	41	26.0	+ 8.3	Mean of 6 catalogues : Washington, Lick, Cape, Greenwich
73	W.Z.C. 227	3	44	33.93	+3.48	+24	8	39.7	+14.8	Yale 1187
78-80	W.Z.C. 350	5	8	16.22	+4.13	+26	22	42.9	+ 9.7	Mean of Yale 2344 and Greenwich 1910, 2221

Telescopes :

C = 24-inch Cassegrainian.

F = 4-inch refractor.

G = 6-inch refractor.

N = 24-inch Newtonian.

R = 8-inch refractor.

Observers :

G = C. C. L. Gregory.

R = Frank Robbins.

Notes

‡ Eye and ear by R. 66 not yet identified.

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